MINT-MERIT

Master of Science in Information and Communication Technologies

COMMUNICATION THEORY

Continuous Time/Amplitude Channels

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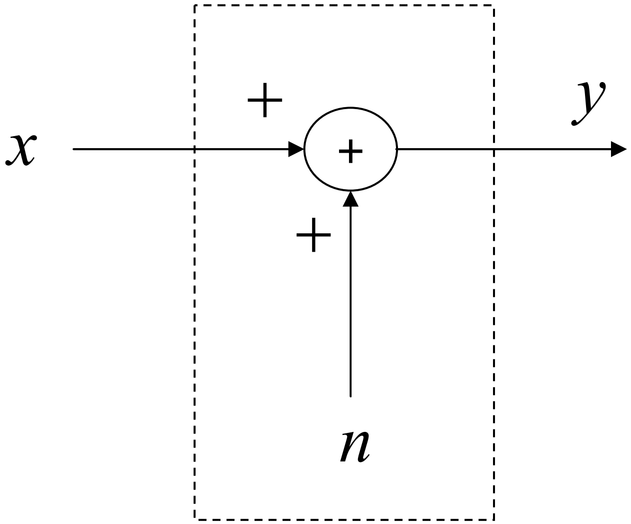
THE ADDITIVE (WHITE) GAUSSIAN CHANNEL: A(W)GC

We will now focus on the following important channel model:



$$x \sim f_x(x)$$

$$n \sim N(0, \sigma_n^{\,2}) \quad \text{with } f_n(n)$$

If 'x' and 'n' independent R.V.:

$$f_y(y) = f_x(\lambda) * f_n(\lambda)\Big|_{\lambda = y}$$

(‘EXTENSION’ OF THE INFORMATION DEFINITION FOR THE CONTINUOUS DISTRIBUTIONS)

Let’s X to be a R.V. with probability density function (p.d.f.) $f_x(X)$

We define entropy H(X) as follows (<u>Differential Entropy</u>)

$$H(X) \equiv \int_{-\infty}^{+\infty} f_x(x) \log \frac{1}{f_x(x)}\, dx \gtrless 0!! \quad \text{and/or arbitrary large}$$

For two R.V. ‘x’ and ‘y’, we also define:

$$H(x;y) = \iint f(x;y) \log \frac{1}{f(x;y)} dxdy$$

$$H(x/y) = \iint f(x;y) \log \frac{1}{f(x/y)} dxdy \qquad \text{and } H(y/x) = \cdots$$

When both H(x) and H(y) are finite, then:

$$H(x;y) \leq H(X) + H(Y)$$

$$H(x;y) = H(X) + H(Y/X) = H(Y) + H(X/Y)$$

$$H(Y/X) \leq H(Y)$$

$$H(X/Y) \leq H(X)$$

All them become equalities (“=“) for ‘x’ and ‘y’ independent.

THEOREM:

For a R.V. with p.d.f. given by $f_x(x)$ and finite variance σ_x^2, the entropy (differential) is upper-bounded by:

$$H(X) \leq \frac{1}{2}\log(2\pi e\sigma_x{}^2)$$

with equality <u>iif</u> 'x' follows a gaussian distribution $x \sim N(\mu;\sigma_x^2)$

This problem follows from the variational solution to:

$$f_x(x) \equiv \arg\max_{f_x(x)} \int_{-\infty}^{+\infty} f_x(x) \log \frac{1}{f_x(x)} dx$$

Subject to:

$$\int_{-\infty}^{+\infty} (x-\mu)^2 f_x(x) dx = \sigma_x{}^2$$

and:

$$\int_{-\infty}^{+\infty} f_x(x) dx = 1 \qquad\qquad f_x(x) \geq 0$$

It is also possible to proceed as follows. We first state that:

$$+\int_{-\infty}^{+\infty} f_x(x)\log\frac{f_x(x)}{g(x)}\,dx \geq 0; \qquad \text{for any p.d.f. } g(x)$$

$$"=" \quad \underline{\text{iif}} \;\; f_x(x) = g(x)\;\forall x$$

We see that:

$$\int_{-\infty}^{+\infty} f_x(x)\log\frac{g(x)}{f_x(x)}\,dx \overset{(*)}{\leq} \frac{1}{\ln 2}\int f_x(x)\left[\frac{g(x)}{f_x(x)}-1\right]dx = 0$$

Thus:

$$H(X) \leq \int_{-\infty}^{+\infty} f_x(x)\log\frac{1}{g(x)}\,dx \qquad\qquad "=" \quad \underline{\text{iif}} \;\; f_x(x) = g(x)$$

We assume ‘x’ and ‘y’ two R.V. with the same mean ‘μ’ and varance σ^2, such that $f_x(x)$ is the p.d.f. for ‘x’ and ‘y’ is a gaussian R.V. $y \sim N(\mu,\sigma^2)$:

We see that:

$$H[Y] = \int f_y(y) \log \frac{1}{f_y(y)} dy = \int f_y(y) \left(\frac{1}{2} \log(2\pi\sigma^2) + \frac{(y-\mu)^2}{2\sigma^2} \right) dy =$$

$$= \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2\sigma^2} E[(y-\mu)^2] = \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2\sigma^2} E[(x-\mu)^2] =$$

$$= \int f_x(x) \log \frac{1}{f_y(x)} dx \geq \int f_x(x) \log \frac{1}{f_x(x)} dx = H\left[X\right]$$

$$\text{"="} \ \underline{\text{iif}} \ \ f_x(x) = f_y(x) \ \ \forall x$$

Then, for:

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2}$$

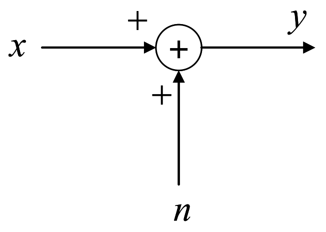
$$H(X) \le \int_{-\infty}^{+\infty} f_x(x) \log \frac{1}{g(x)} dx =$$

$$= \frac{1}{2}\log(2\pi\sigma_x{}^2) + \frac{\log e}{2\sigma_x{}^2} \underbrace{\int_{-\infty}^{+\infty} (x-\mu)^2 f_x(x) dx}_{\sigma_x{}^2} =$$

$$= \frac{1}{2}\log(2\pi e\sigma_x{}^2) \;\; !!$$

$$"=" \quad \underline{\text{iif}} \;\; f_x(x) = g(x)\ \forall x$$

CAPACITY FOR A GAUSSIAN CHANNEL



$$n \sim N(0, \sigma_n^2)$$

$$I(A_x; A_y) = H(Y) - H(Y/X)$$

- For a given 'x':

$$y \sim N(x, \sigma_n^2) \Rightarrow H(Y/X) = \frac{1}{2}\log(2\pi e \sigma_n^2)$$

For H(Y), we have that: $y = x + n$ with 'x'; 'n' independent R.V.

thus: $\sigma_y^2 = \sigma_x^2 + \sigma_n^2$

Finally:

$$C \equiv \max_{f_x(x)} I(A_x; A_y) \Rightarrow \text{if} \quad \left. \begin{array}{l} x \sim \mathrm{N}(\mu; \sigma_x^2) \\ n \sim \mathrm{N}(0; \sigma_n^2) \end{array} \right| \Rightarrow y = x + n \sim \mathrm{N}(\mu; \sigma_x^2 + \sigma_n^2)$$

$$C = \underbrace{\frac{1}{2}\log(2\pi e(\sigma_x^2 + \sigma_n^2))}_{H(Y)} - \underbrace{\frac{1}{2}\log(2\pi e \sigma_n^2)}_{H(Y/X)} = \frac{1}{2}\log\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right) \quad !!!$$

Thus, for a discrete-time gaussian channel:

$$C = \frac{1}{2}\log\left(1+\frac{\sigma_x^{\ 2}}{\sigma_n^{\ 2}}\right) = \frac{1}{2}\log(1+SNR) \quad \text{with : } SNR \equiv \frac{\sigma_x^{\ 2}}{\sigma_n^{\ 2}}$$

If the channel is a band-limited channel in the interval (-BW,+BW) and for an additive WHITE gaussian noise (*AWGN*):

$$\sigma_n^{\ 2} = \frac{N_o}{2} 2BW = N_o BW$$

$$\left(S_{nn}(f) = \frac{N_o}{2} \quad watts / Hz\right)$$

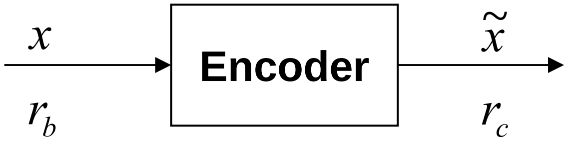
$$(e^{j2\pi f})$$

$$C = \frac{1}{2}\log\left(1+\frac{\sigma_x^{\ 2}}{N_o BW}\right)$$

For a transmission rate r_b bits/sec, we can also define the "*Energy per Information Bit*" or E_b as follows:

$$E_b \equiv \sigma_x^{\ 2} T_b \equiv \sigma_x^{\ 2} / r_b \quad \Rightarrow \quad C = \frac{1}{2}\log\left(1+\frac{E_b r_b}{N_o BW}\right)$$

Impact of the encoder:



$$R_c \equiv \frac{r_b}{r_c} \quad \text{(code-rate)}$$

Minimum transmission bandwidth (Nyquist Bandwidth): $BW = \frac{r_c}{2} = \frac{r_b}{2R_c}$

Thus: $$C = \frac{1}{2}\log\left(1 + 2R_c \frac{E_b}{N_o}\right) \ \text{bits}/\text{channel use}$$

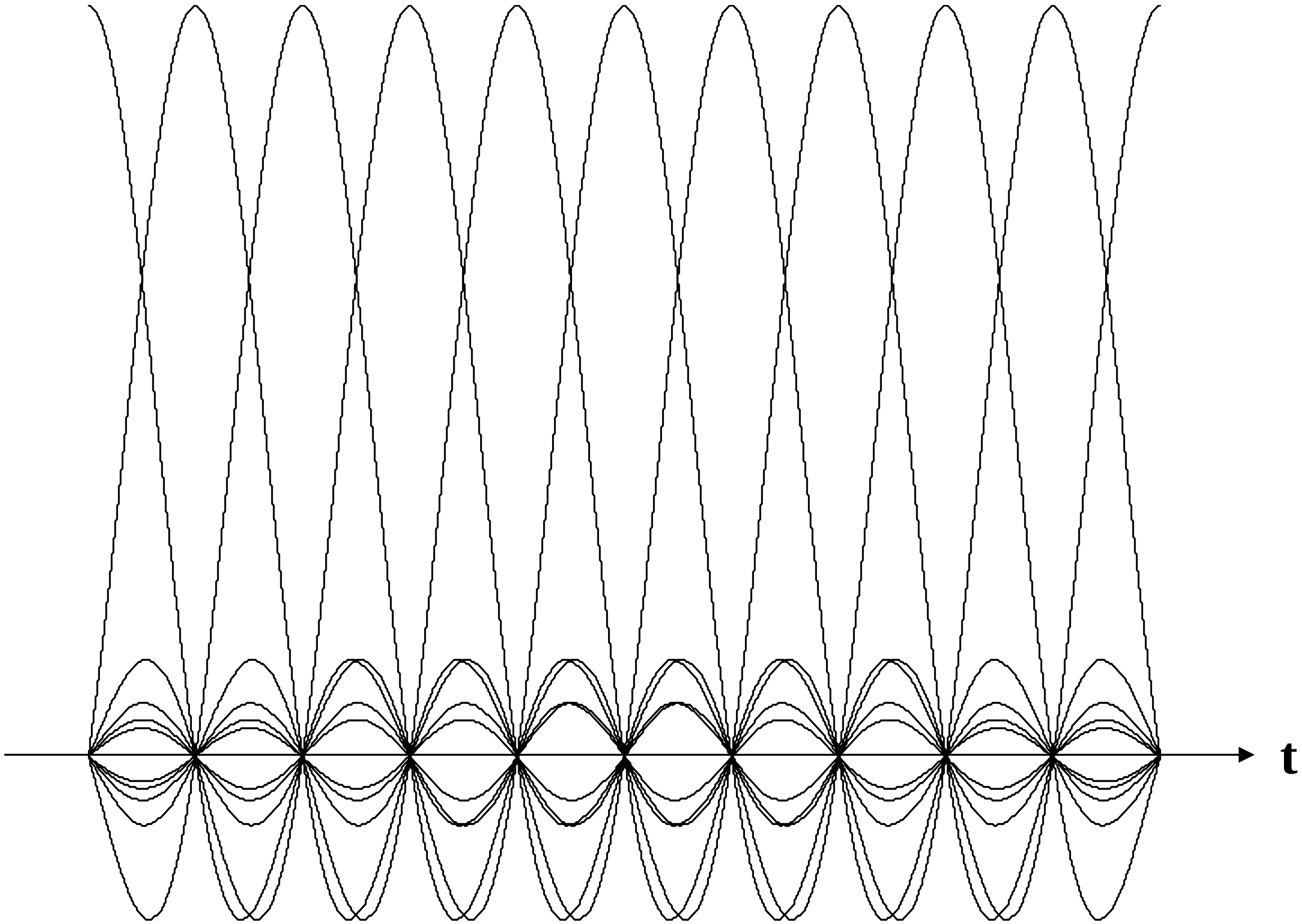
For a channel with bandwidth BW (Hz), a total of 2BW channel-uses/sec are possible still preserving the DMC nature of the channel [(!)]:

$$C = BW\log(1+SNR) \overset{(*)}{=} BW\log\left(1+\frac{\sigma_x^2}{N_o BW}\right) = BW\log\left(1+\frac{E_b r_b}{N_o BW}\right) =$$

$$= BW\log\left(1+2R_c\frac{E_b}{N_o}\right) \ \text{bits}/\text{sec}$$

(*) White noise

(!) Moreover, 2BW samples/sec suffices for a perfect time representation of a band-limited signal of bandwidth BW (Hz).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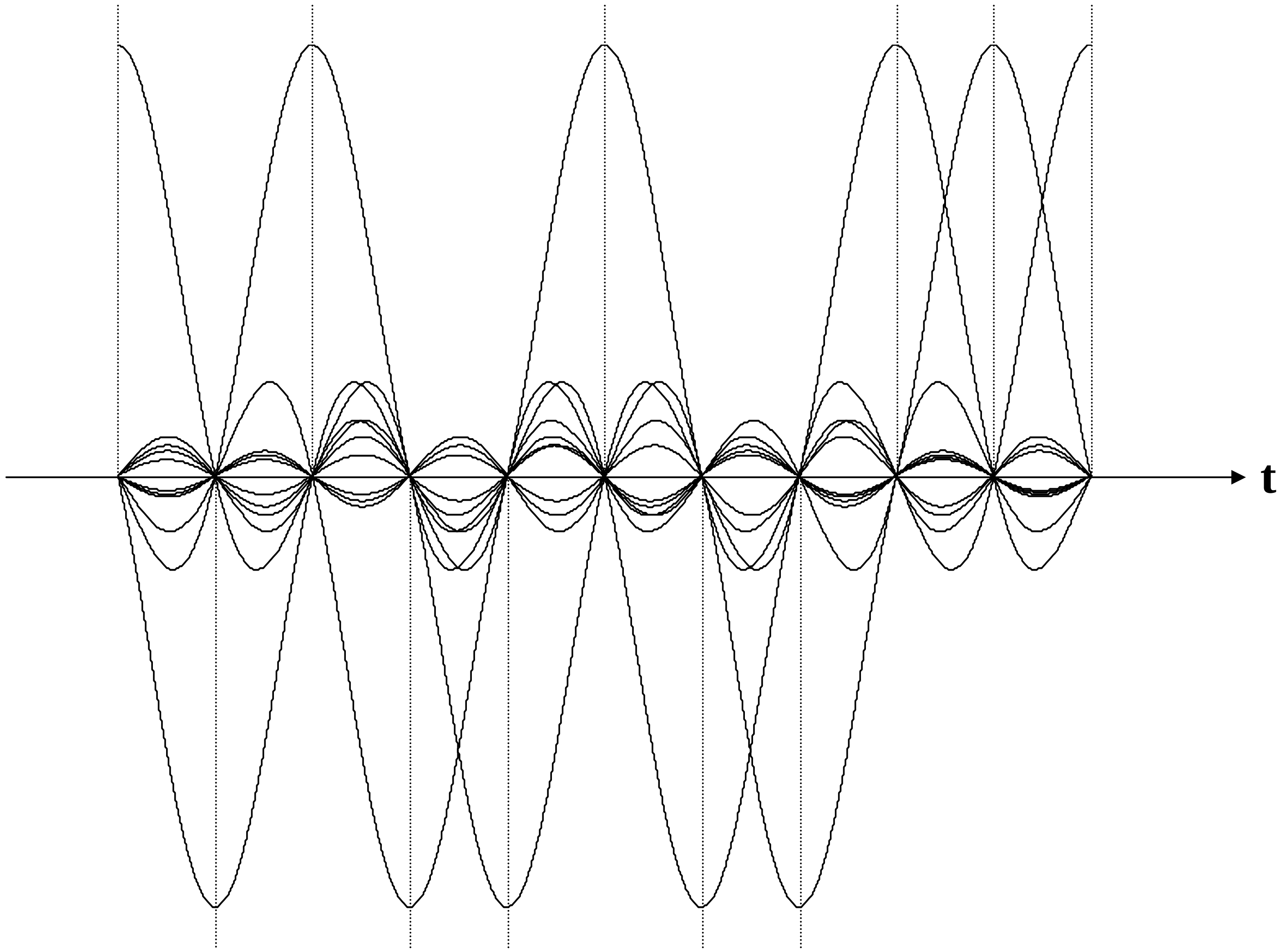
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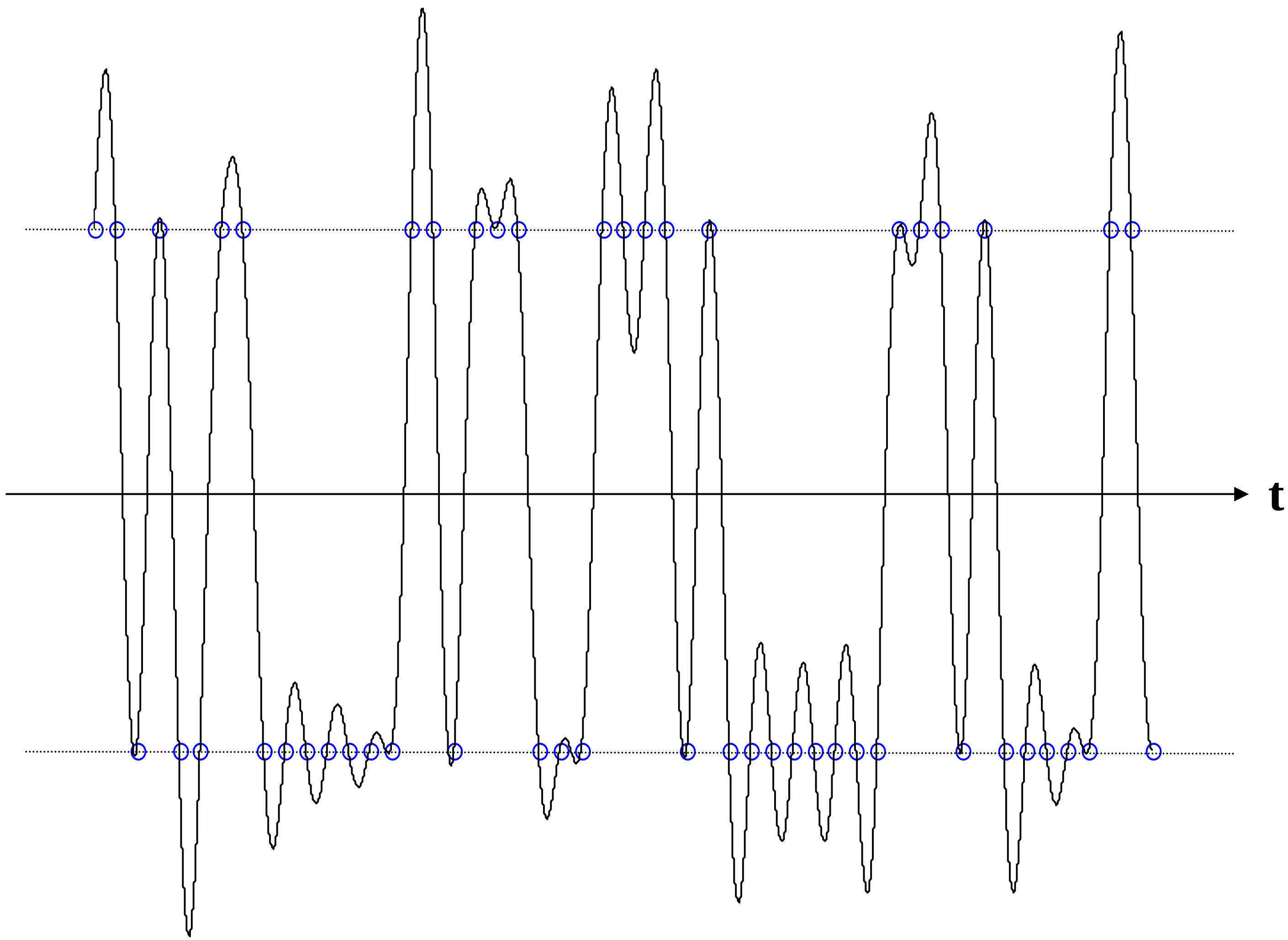
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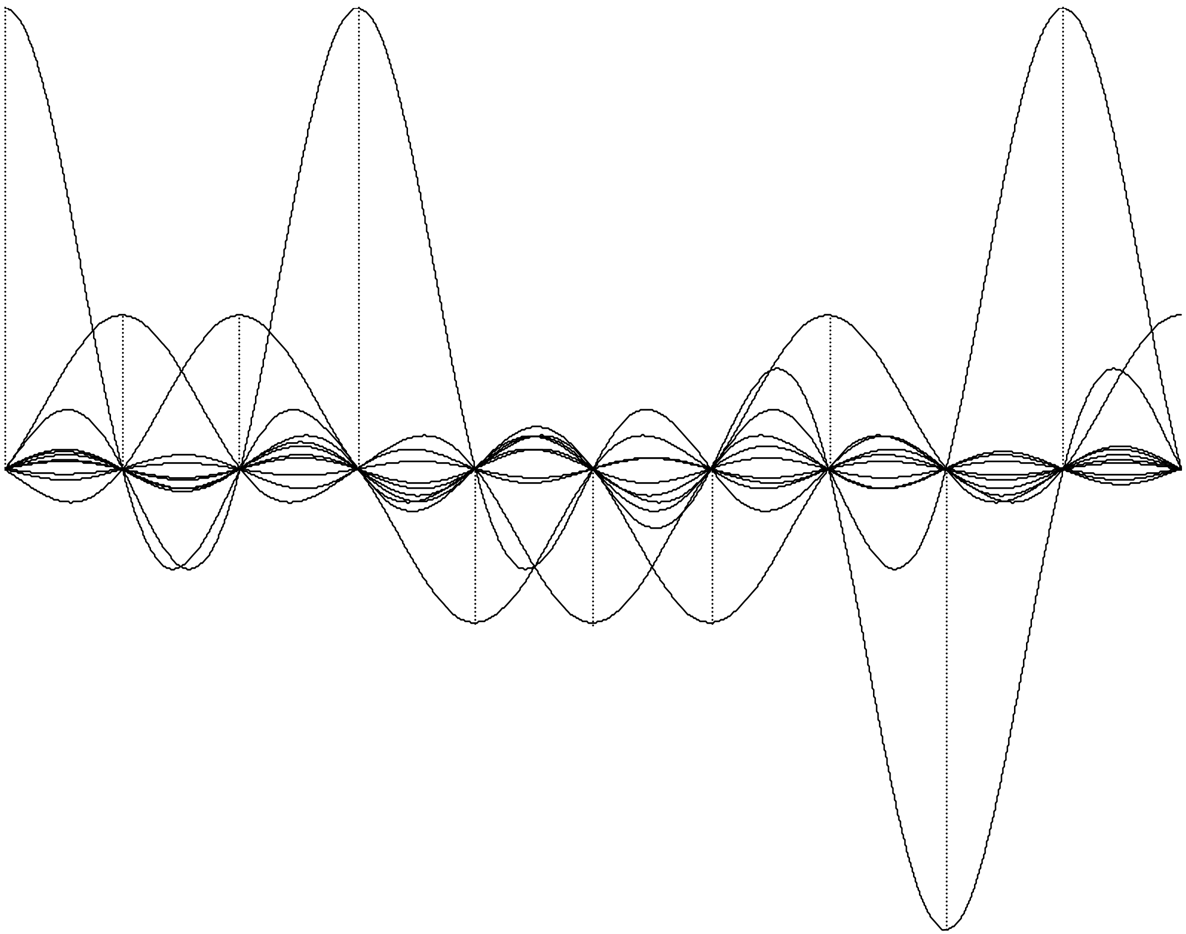
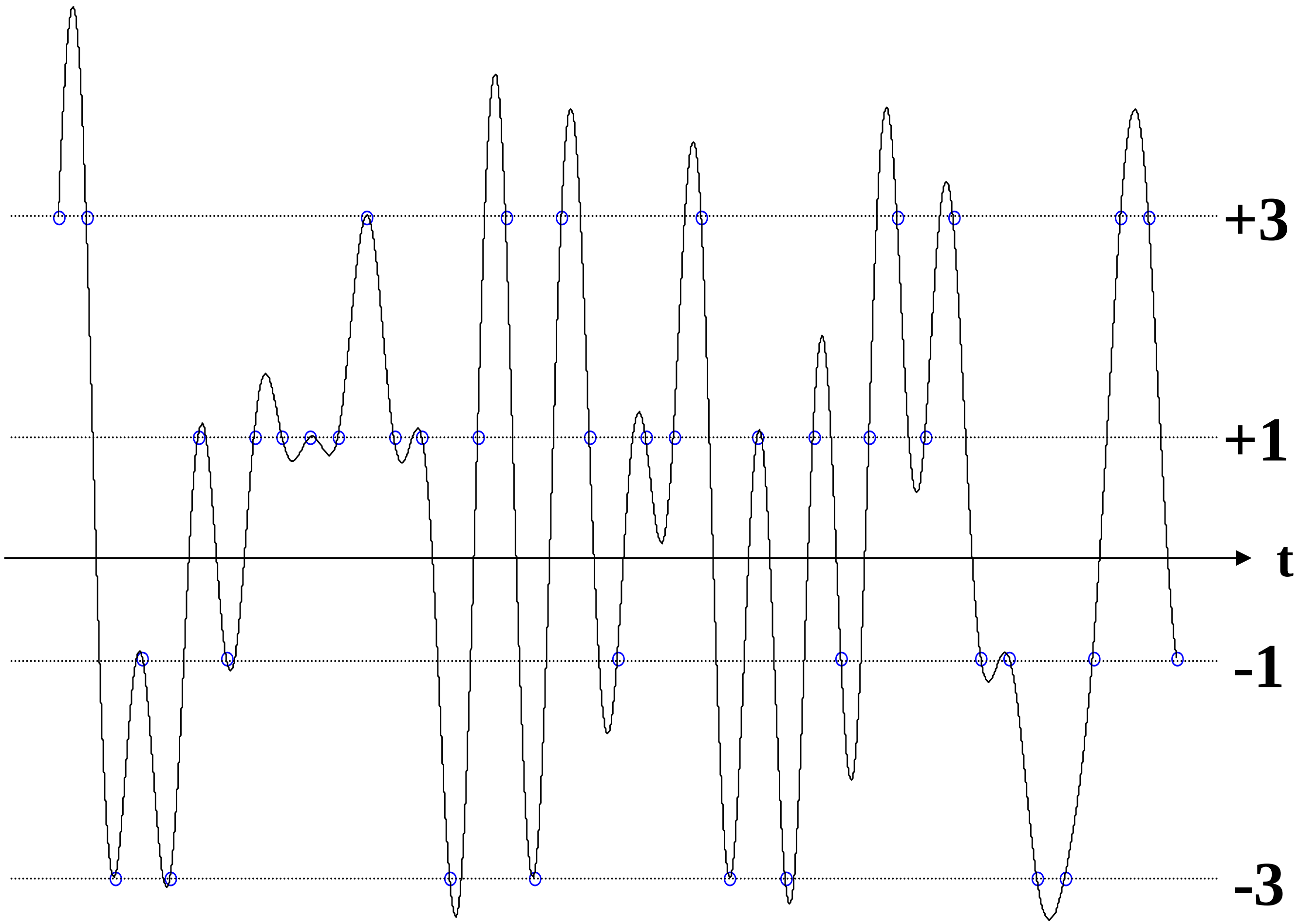



+3
+1
t
-1
-3

Finally, for an efficient code (R_c=1) we can relax the bandwidth constraint:

$$\lim_{BW\to\infty} C \longrightarrow \lim_{BW\to\infty} \log\left(1+\frac{E_b r_b}{N_o BW}\right)^{BW} = \lim_{BW\to\infty} \log\underbrace{\left(1+\frac{1}{BWN_o / E_b r_b}\right)^{\frac{BWN_o}{E_b r_b}}}_{e}{}^{\cdot\frac{E_b r_b}{N_o}} \rightarrow$$

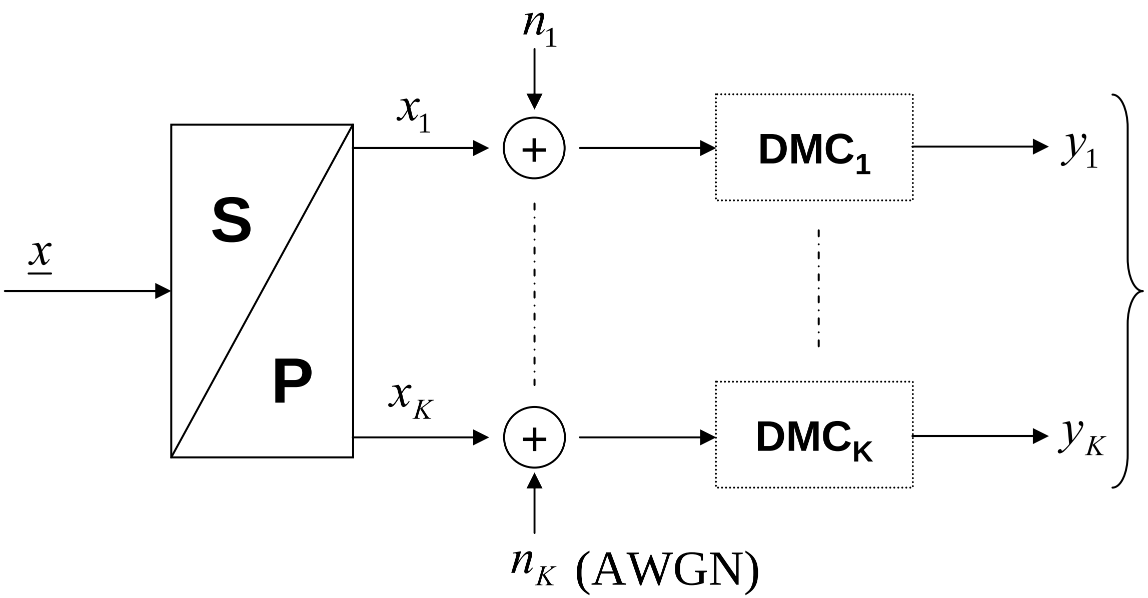
$$\rightarrow \frac{E_b r_b}{N_o}\log e$$

$$\boxed{\lim_{BW\to\infty} C \rightarrow \frac{E_b r_b}{N_o}\log e} \quad \text{bits}/\text{sec}$$

Thus:

$$r_b \le C \Rightarrow r_b \le \frac{E_b r_b}{N_o}\log e \Rightarrow \frac{E_b}{N_o} \ge \frac{1}{\log e} \cong \underline{\underline{-1.59dB!!}}$$

$$\text{(Spectral Efficiency (E}_r\text{) is defined as:} \quad E_r \equiv \frac{r_b}{BW} \quad \text{bits/sec/Hz)}$$



$$C_i = BW_i \log(1 + SNR_i)$$

with: $$SNR_i \equiv \frac{\sigma_{x_i}^2}{\sigma_{n_i}^2}$$

Because the independence + the DMC condition:

$$H(\underline{Y}) = \sum_i H(Y_i)$$

$$H(\underline{Y} / \underline{X}) = \sum_i H(Y_i / X_i)$$

$$C = \sum_i \underbrace{BW_i \log(1 + SNR_i)}_{C_i} = \log \prod_{i=1}^{K} (1 + SNR_i)^{BW_i} \quad \ldots$$

(Consequences!)

THEOREM:

The capacity for ‘K’ parallel additive gaussian channels under a total transmitted power S_T constraint is given by:

$$C(S_T) = \sum_{k=1}^{K} \frac{1}{2} \log\left(1 + \frac{\sigma_i^{\,2}}{\sigma_{ni}^{\,2}}\right)$$

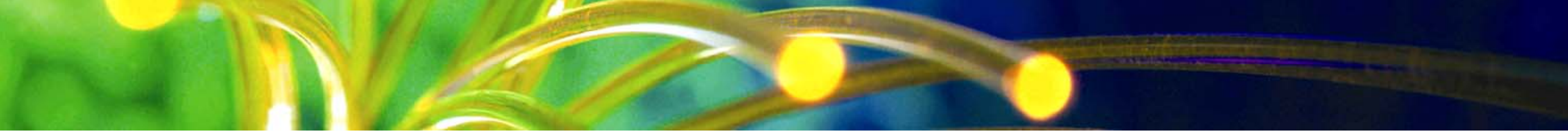
With:

$$\Rightarrow \sigma_i^{\,2} = \max(0; \lambda - \sigma_i^{\,2})$$

$$\Rightarrow '\lambda' \text{ such that } \sum_{k=1}^{K} \sigma_i^{\,2} = S_T \longrightarrow$$

“WATER-POURING” concept

This capacity is achieved if the channels are gaussian-independent.



1.- We saw that:

$$I(\underline{\mathbf{x}};\underline{\mathbf{y}}) \le \sum_{k=1}^{K} I(x_k; y_k) \qquad \text{"="} \ \underline{\text{if}} \ \text{independent (sufficient)}$$

$$\downarrow$$

$$\frac{1}{2}\log\left(1+\frac{\sigma_i^{\,2}}{\sigma_{ni}^{\,2}}\right) \quad i = 1,2,\ldots,K$$

2.- From the Lagrange Multiplier technique:

$$\max_{\{\sigma_k^{\,2}\}} \quad \sum_{k=1}^{K}\frac{1}{2}\log\left(1+\frac{\sigma_k^{\,2}}{\sigma_{n_k}^{\,2}}\right)$$

Subject to:

$$\sum_{k=1}^{K}\sigma_k^{\,2} = S_T$$

$$\equiv \max_{\{\sigma_k^{\,2}\}} \underbrace{\sum_{k=1}^{K}\frac{1}{2}\log\left(1+\frac{\sigma_k^{\,2}}{\sigma_{nk}^{\,2}}\right) + \mu\left[\sum_{k=1}^{K}\sigma_k^{\,2} - S_T\right]}_{\Phi\left[\underline{\sigma}^2\right]}$$

Thus:

$$\frac{\partial}{\partial \sigma_i^{\ 2}} \Phi(\underline{\sigma}^2) = \frac{1}{2} \cdot \frac{1 / \sigma_{n_i}^{\ 2}}{1 + \sigma_i^{\ 2} / \sigma_{n_i}^{\ 2}} + \mu = \frac{1}{2} \cdot \frac{1}{\sigma_i^{\ 2} + \sigma_{n_i}^{\ 2}} + \mu = 0$$

That is:

$$\sigma_i^{\ 2} = -\frac{1}{2\mu} - \sigma_{n_i}^{\ 2} \geq 0 \ \ !! \qquad \Rightarrow \qquad \boxed{\sigma_i^{\ 2} = \lambda - \sigma_{n_i}^{\ 2} \geq 0}$$

$$\text{For: } \quad \lambda = -\frac{1}{2\mu}$$

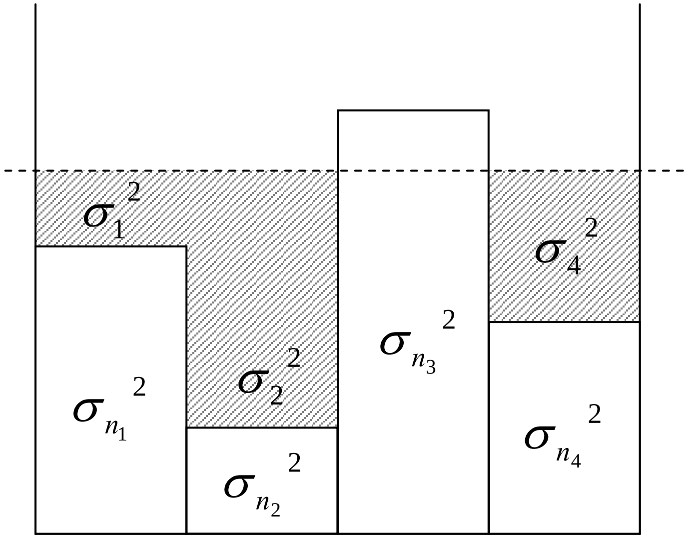
$$\lambda = -\frac{1}{2\mu} = \left[S_T + \sum_{k=1}^{K_0} \sigma_{n_k}^{\ 2} \right] \frac{1}{K_o}$$

$$K_o = \sum_{n=1}^{K} K_n$$

$$K_n = \begin{cases} 1 & \text{if} \quad \lambda - \sigma_{\mathrm{n_k}}^{\ 2} > 0 \\ 0 & \text{if} \quad \lambda - \sigma_{\mathrm{n_k}}^{\ 2} \leq 0 \end{cases}$$

$$\boxed{\sigma_i^{\ 2} = 0 \quad \text{for} \quad \lambda - \sigma_{ni}^{\ 2} < 0}$$

Water-Pouring (Water-Filling):



$$\boxed{S_T = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}$$

$$\sigma_i^2 = \max(0; \lambda - \sigma_{ni}^2)$$

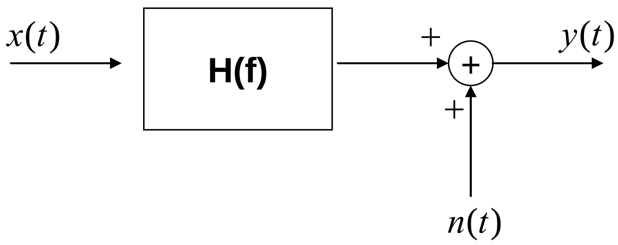
$$\lambda = \frac{1}{K_o}\left[S_T + \sum_{k=1}^{K_0} \sigma_{nk}^2\right]$$

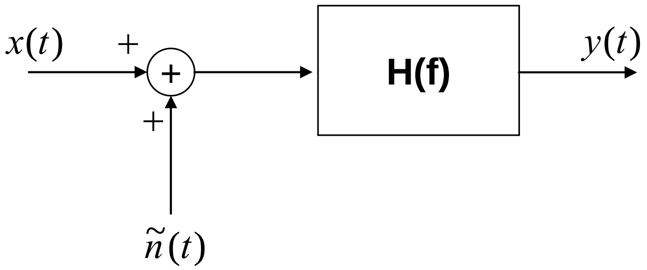
MORE GENERAL CASES:

1.- Define the "signal to noise spectral density ratio"

$$SNR(f) \equiv \frac{S_{xx}(f)}{S_{nn}(f)} \;\Rightarrow\; C = \int_{BW} \log\left(1 + \frac{S_{xx}(f)}{S_{nn}(f)}\right) df$$

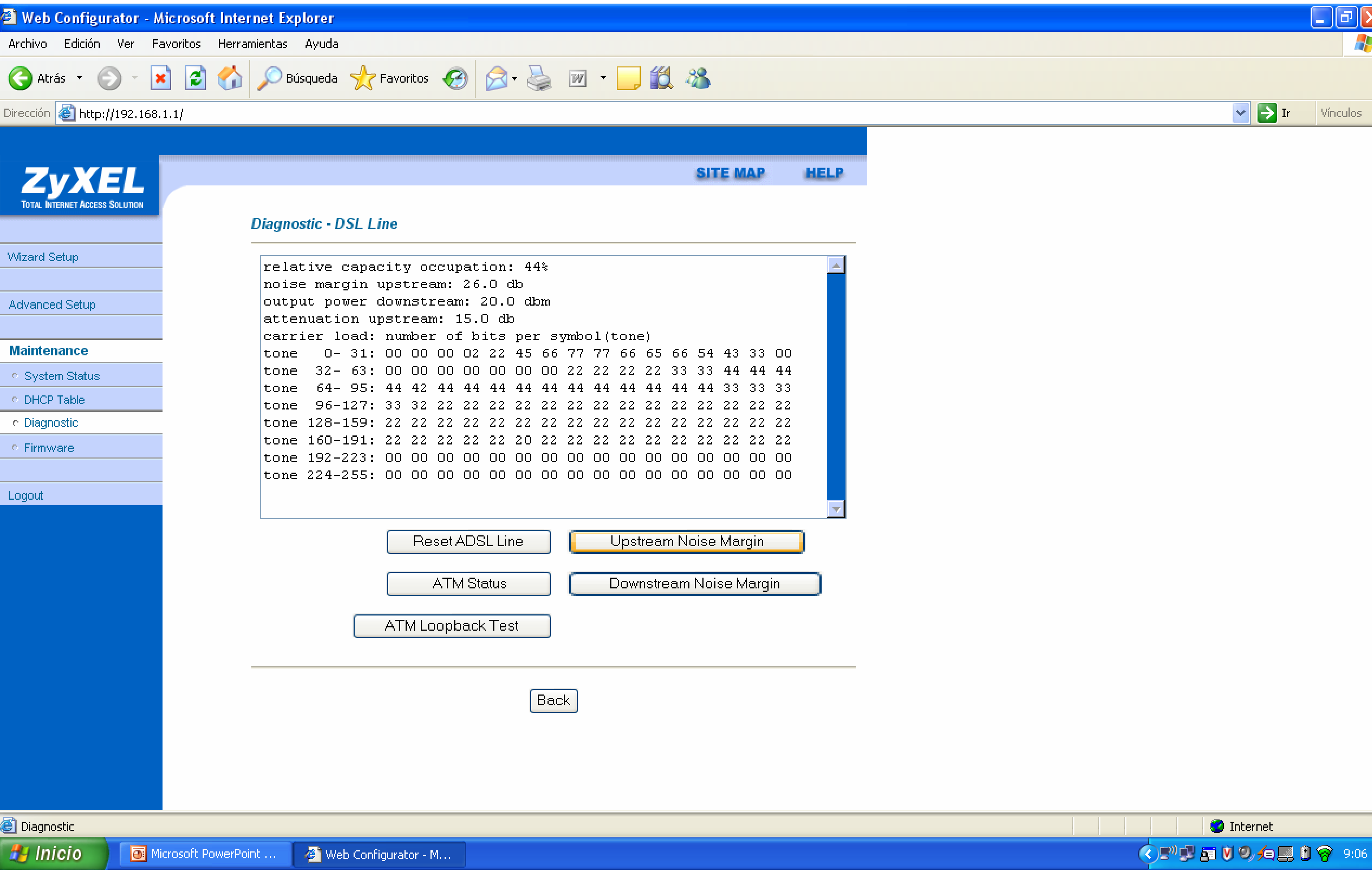
2.- Frequency-selective channels (channels with memory)





(think on the impact of the channel memory…)

$$S_{\tilde{n}\tilde{n}}(f) = \frac{S_{nn}(f)}{|H(f)|2}$$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Atrás Búsqueda Favoritos
Dirección http://192.168.1.1/ Ir Vínculos
ZyXEL
TOTAL INTERNET ACCESS SOLUTION
SITE MAP HELP
Wizard Setup
Advanced Setup
Maintenance
System Status
DHCP Table
Diagnostic
Firmware
Logout
Diagnostic - DSL Line
relative capacity occupation: 44%
noise margin upstream: 26.0 db
output power downstream: 20.0 dbm
attenuation upstream: 15.0 db
carrier load: number of bits per symbol(tone)
tone 0- 31: 00 00 00 02 22 45 66 77 77 66 65 66 54 43 33 00
tone 32- 63: 00 00 00 00 00 00 00 22 22 22 22 33 33 44 44 44
tone 64- 95: 44 42 44 44 44 44 44 44 44 44 44 44 44 33 33 33
tone 96-127: 33 32 22 22 22 22 22 22 22 22 22 22 22 22 22 22
tone 128-159: 22 22 22 22 22 22 22 22 22 22 22 22 22 22 22 22
tone 160-191: 22 22 22 22 22 20 22 22 22 22 22 22 22 22 22 22
tone 192-223: 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
tone 224-255: 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
Reset ADSL Line
Upstream Noise Margin
ATM Status
Downstream Noise Margin
ATM Loopback Test
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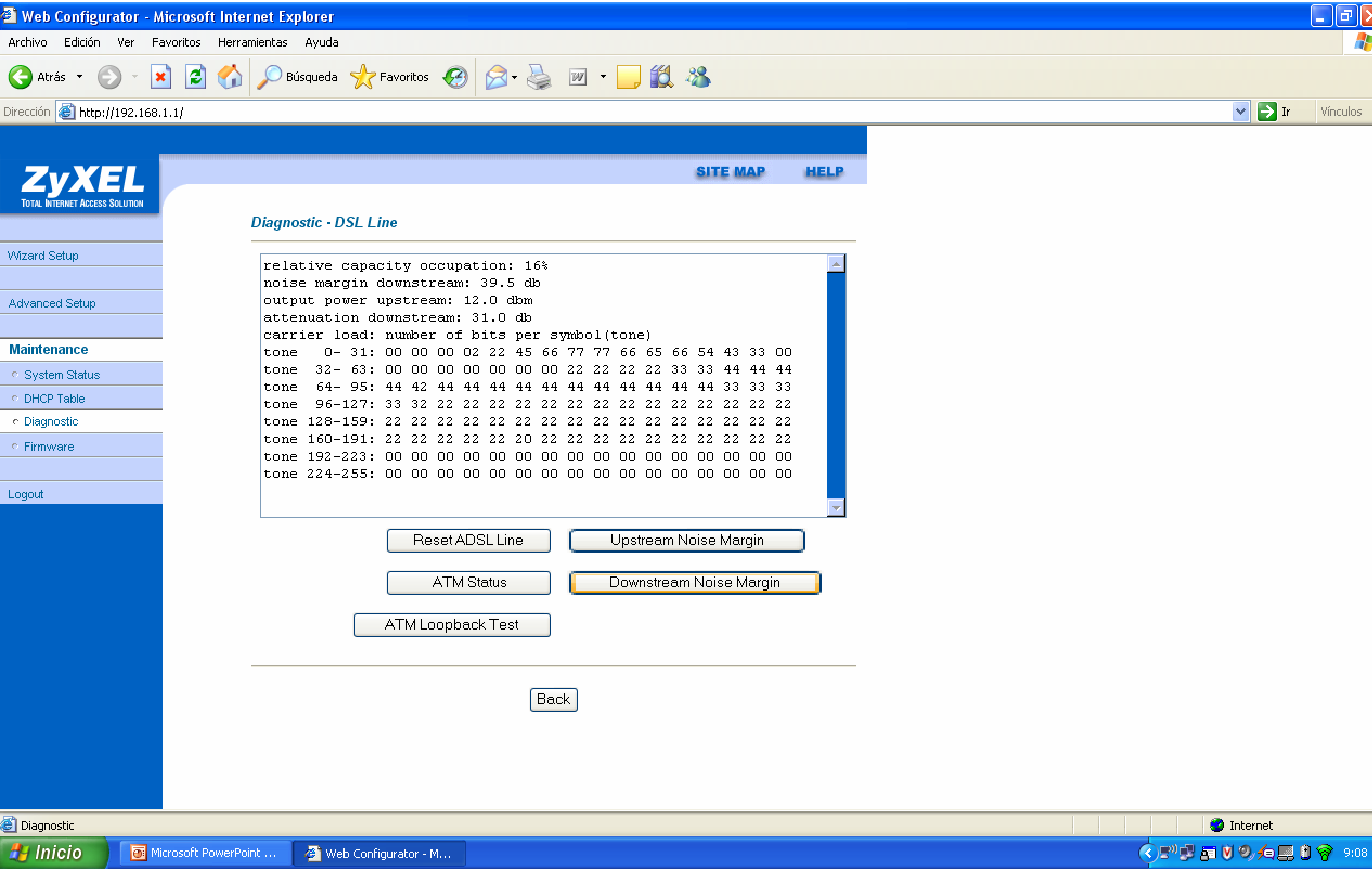
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TOTAL INTERNET ACCESS SOLUTION
SITE MAP HELP
Wizard Setup
Advanced Setup
Maintenance
System Status
DHCP Table
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Firmware
Logout
Diagnostic - DSL Line
relative capacity occupation: 16%
noise margin downstream: 39.5 db
output power upstream: 12.0 dbm
attenuation downstream: 31.0 db
carrier load: number of bits per symbol(tone)
tone 0- 31: 00 00 00 02 22 45 66 77 77 66 65 66 54 43 33 00
tone 32- 63: 00 00 00 00 00 00 00 22 22 22 22 33 33 44 44 44
tone 64- 95: 44 42 44 44 44 44 44 44 44 44 44 44 44 33 33 33
tone 96-127: 33 32 22 22 22 22 22 22 22 22 22 22 22 22 22 22
tone 128-159: 22 22 22 22 22 22 22 22 22 22 22 22 22 22 22 22
tone 160-191: 22 22 22 22 22 20 22 22 22 22 22 22 22 22 22 22
tone 192-223: 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
tone 224-255: 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
Reset ADSL Line
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Downstream Noise Margin
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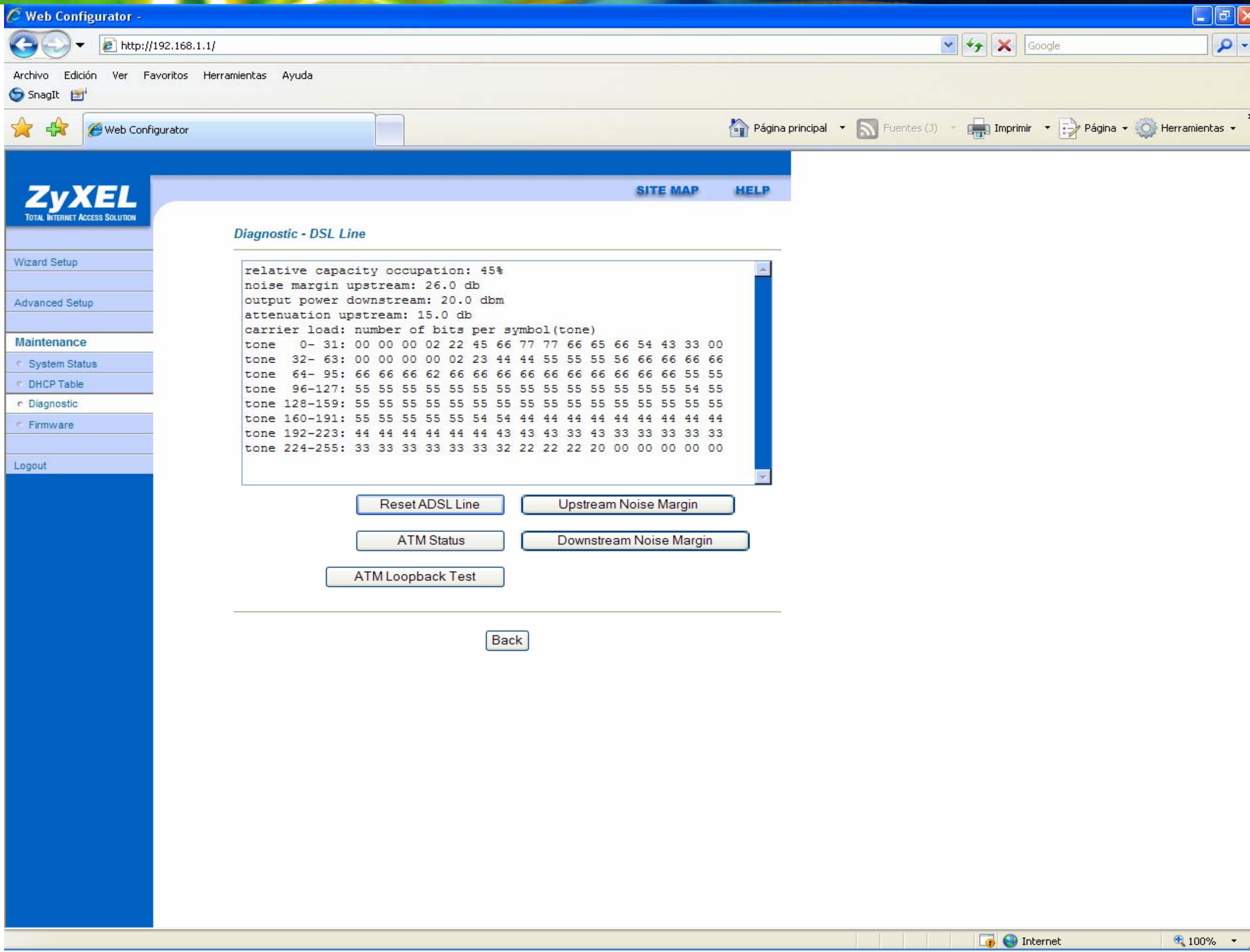
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http://192.168.1.1/
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Página
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TOTAL INTERNET ACCESS SOLUTION
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Wizard Setup
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System Status
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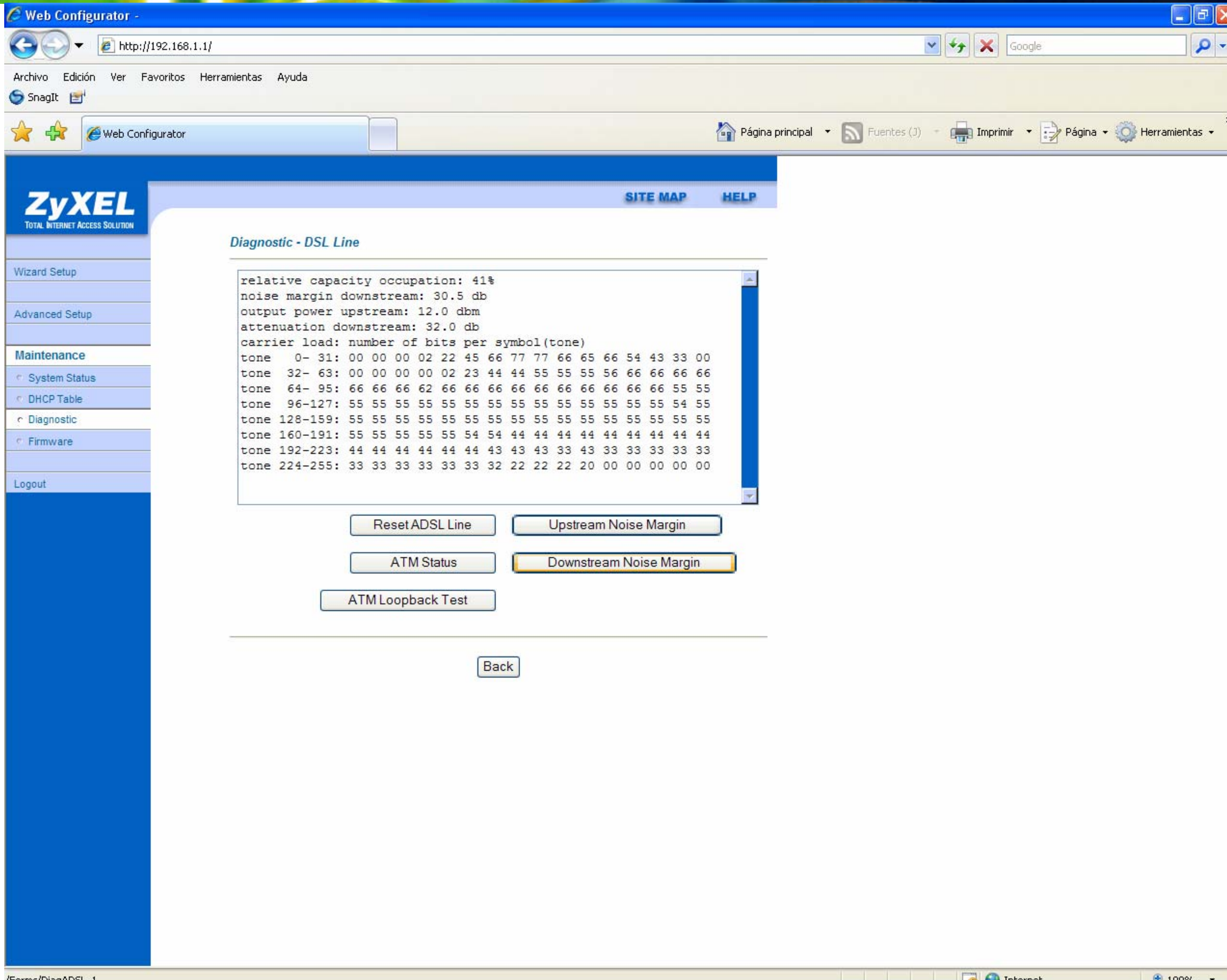
Logout
Diagnostic - DSL Line
relative capacity occupation: 45%
noise margin upstream: 26.0 db
output power downstream: 20.0 dbm
attenuation upstream: 15.0 db
carrier load: number of bits per symbol(tone)
tone 0- 31: 00 00 00 02 22 45 66 77 77 66 65 66 54 43 33 00
tone 32- 63: 00 00 00 00 02 23 44 44 55 55 55 56 66 66 66 66
tone 64- 95: 66 66 66 62 66 66 66 66 66 66 66 66 66 66 55 55
tone 96-127: 55 55 55 55 55 55 55 55 55 55 55 55 55 55 54 55
tone 128-159: 55 55 55 55 55 55 55 55 55 55 55 55 55 55 55 55
tone 160-191: 55 55 55 55 55 54 54 44 44 44 44 44 44 44 44 44
tone 192-223: 44 44 44 44 44 44 43 43 43 33 43 33 33 33 33 33
tone 224-255: 33 33 33 33 33 33 32 22 22 22 20 00 00 00 00 00
Reset ADSL Line
Upstream Noise Margin
ATM Status
Downstream Noise Margin
ATM Loopback Test
Back
Internet
100%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TOTAL INTERNET ACCESS SOLUTION
SITE MAP HELP
Wizard Setup
Advanced Setup
Maintenance
System Status
DHCP Table
Diagnostic
Firmware
Logout
Diagnostic - DSL Line
relative capacity occupation: 41%
noise margin downstream: 30.5 db
output power upstream: 12.0 dbm
attenuation downstream: 32.0 db
carrier load: number of bits per symbol(tone)
tone 0- 31: 00 00 00 02 22 45 66 77 77 66 65 66 54 43 33 00
tone 32- 63: 00 00 00 00 02 23 44 44 55 55 55 56 66 66 66 66
tone 64- 95: 66 66 66 62 66 66 66 66 66 66 66 66 66 66 55 55
tone 96-127: 55 55 55 55 55 55 55 55 55 55 55 55 55 55 54 55
tone 128-159: 55 55 55 55 55 55 55 55 55 55 55 55 55 55 55 55
tone 160-191: 55 55 55 55 55 54 54 44 44 44 44 44 44 44 44 44
tone 192-223: 44 44 44 44 44 44 43 43 43 33 43 33 33 33 33 33
tone 224-255: 33 33 33 33 33 33 32 22 22 22 20 00 00 00 00 00
Reset ADSL Line
Upstream Noise Margin
ATM Status
Downstream Noise Margin
ATM Loopback Test
Back
/Forms/DiagADSL_1
Internet
100%