MINT-MERIT

Master of Science in Information and Communication Technologies

COMMUNICATION THEORY

Performance Bounds

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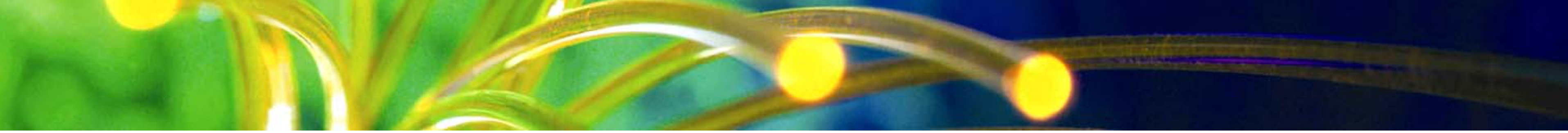


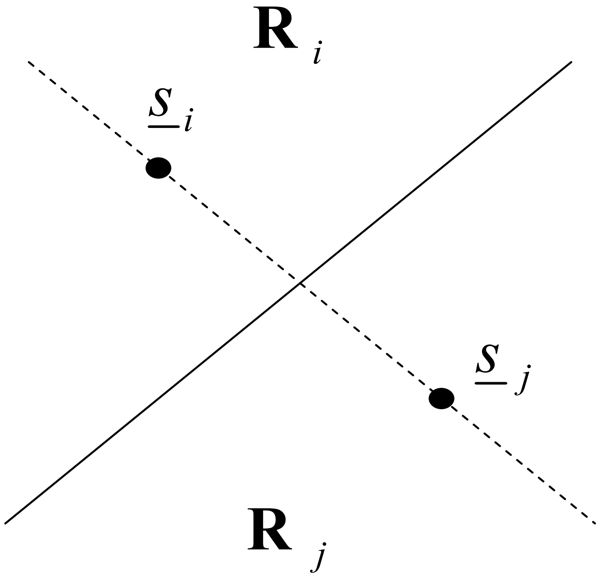
Readings:

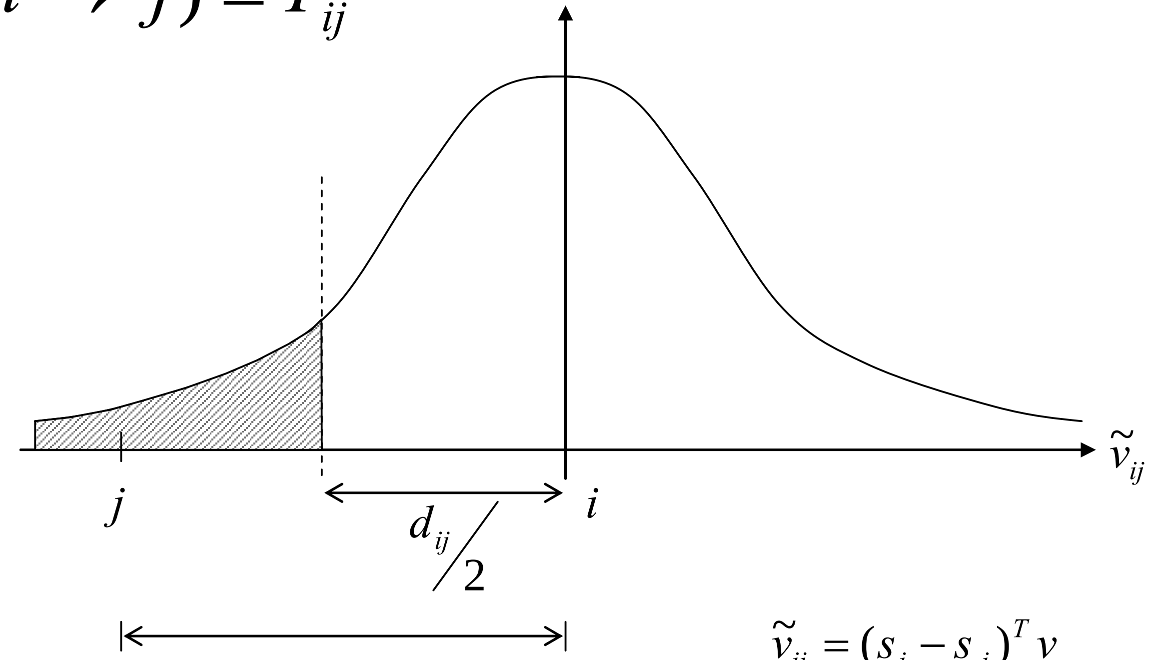
[1] “Principles of Digital Communications and Coding”
A.J. Viterbi – J.K. Omura,
McGraw-Hill -1979
Chapter two, pp. 47-127.

[2] “Principles of Digital Transmission”
S. Benedetto – E. Biglieri,
Kluwer Acad./Plenum PWB – 1999
Chapter five, pp. 215-264.
Chapter six, pp. 272-310.

PAIRWISE ERROR PROBABILITY

$$P_e(i \to j) \equiv P_{ij}^e$$



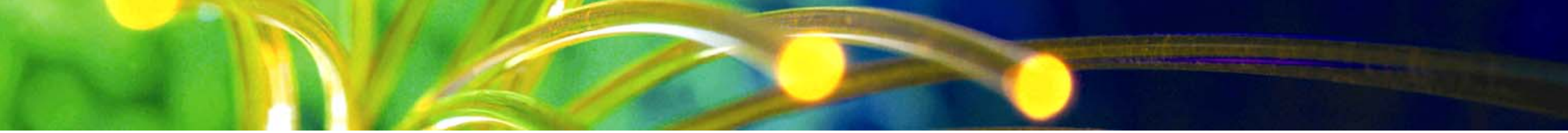


$$\left\|\underline{s}_i - \underline{s}_j\right\| = d_{ij}$$

$$\tilde{v}_{ij} = (\underline{s}_i - \underline{s}_j)^T \underline{v}$$

$$\boxed{P_{ij}^e = Q\left(\frac{\left\|\underline{s}_i - \underline{s}_j\right\|}{\sqrt{2N_o}}\right)}$$

Where $Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\lambda^2/2} \, d\lambda$



UNION-BOUND

We know that: $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

Inequality that can be generalized (induction) to: $P\left(\bigcup_{i=1}^{M} A_i\right) \leq \sum_{i=1}^{M} P(A_i)$

Thus, the probability of error is given by:

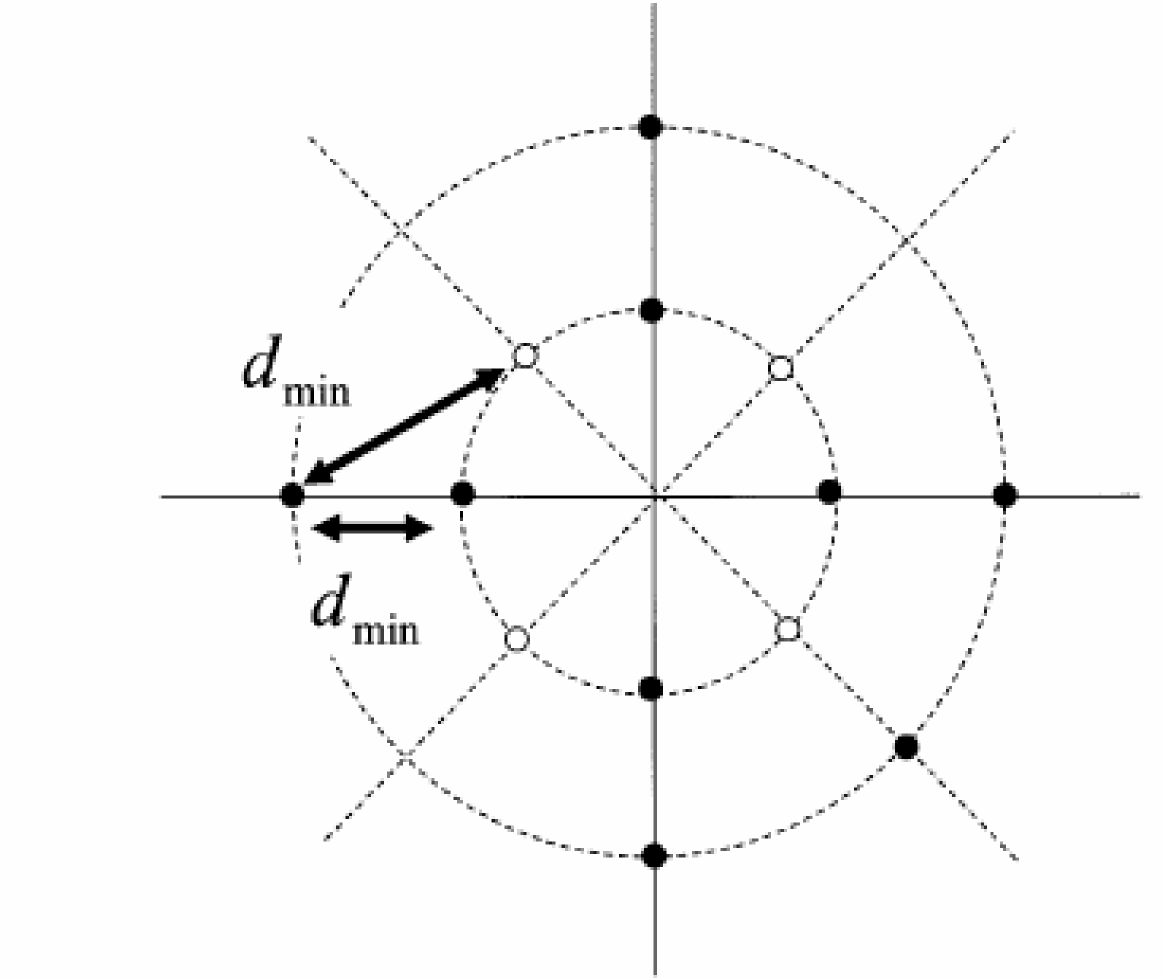
$$\boxed{P(e) = \sum_{m=1}^{M} P(e / \underline{s}_m) P(\underline{s}_m)}$$

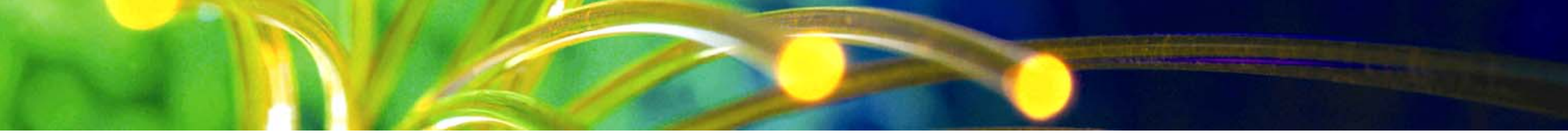
Can be upper-bounded by:

$$P(e / \underline{s}_m) = \Pr(\underline{v} \notin \mathbf{R}_m / \underline{s}_m) \leq \sum_{\substack{m=1 \\ m \neq k}}^{M} P_e(m \rightarrow k) = \sum_{\substack{m=1 \\ m \neq k}}^{M} P_{mk}^{e}$$

$$\boxed{P(e) \leq \sum_{m=1}^{M} \sum_{\substack{k=1 \\ k \neq m}}^{M} P_{mk}^{e} P(\underline{s}_m) \quad \text{with} \quad P_{mk}^{e} = Q\left(\frac{\left\|\underline{s}_m - \underline{s}_k\right\|}{\sqrt{2N_o}}\right)}$$

The “minimum distance” concept of a constellation





If we define the *minimum distance* or d$_{\min}$ as follows:

$$d_{ij} \equiv \left\| \underline{s}_i - \underline{s}_j \right\| \longrightarrow \boxed{d_{\min} \equiv \min_{i,j} \left\{ d_{ij} \right\}_{\substack{1 \le i \le M \\ 1 \le j \le M \\ i \ne j}}}$$

It is possible to see that:

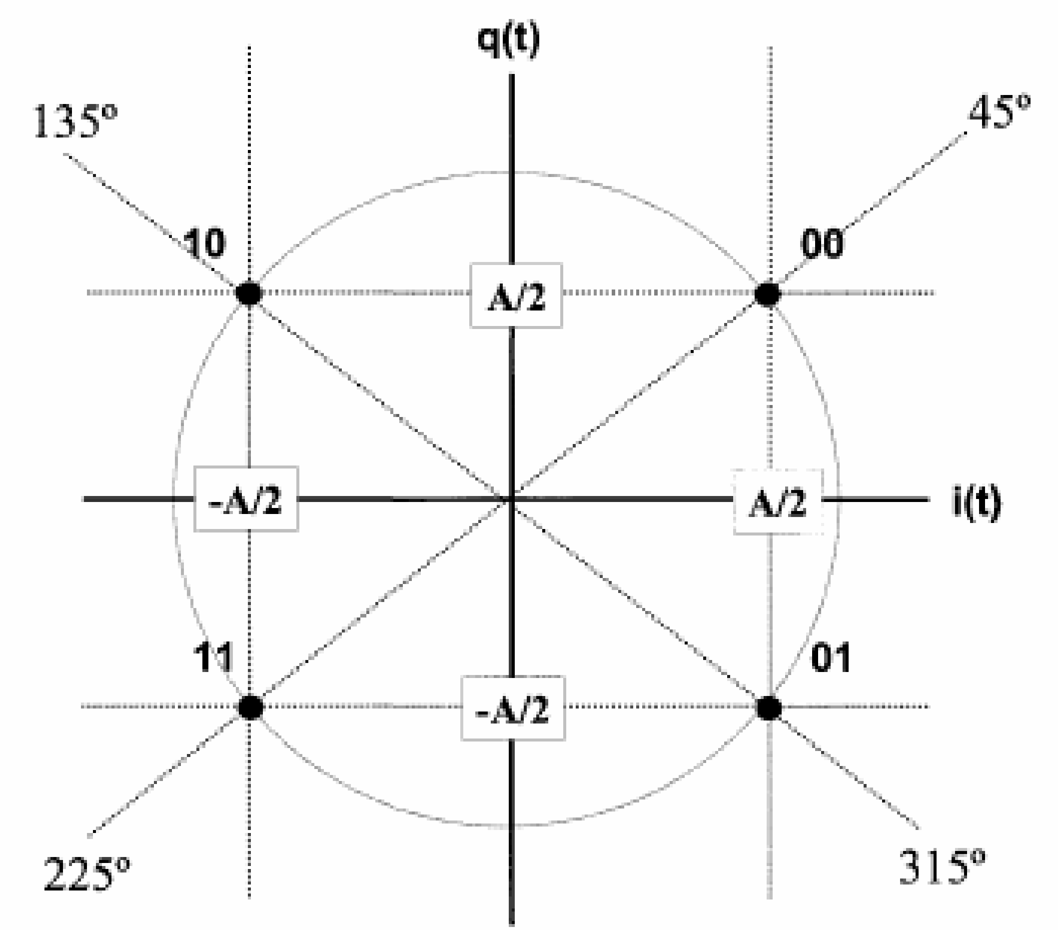
$$P_{mk}^{e} = Q\left(\frac{\left\| \underline{s}_m - \underline{s}_k \right\|}{\sqrt{2N_o}} \right) \le Q\left(\frac{d_{\min}}{\sqrt{2N_o}} \right)$$

and:

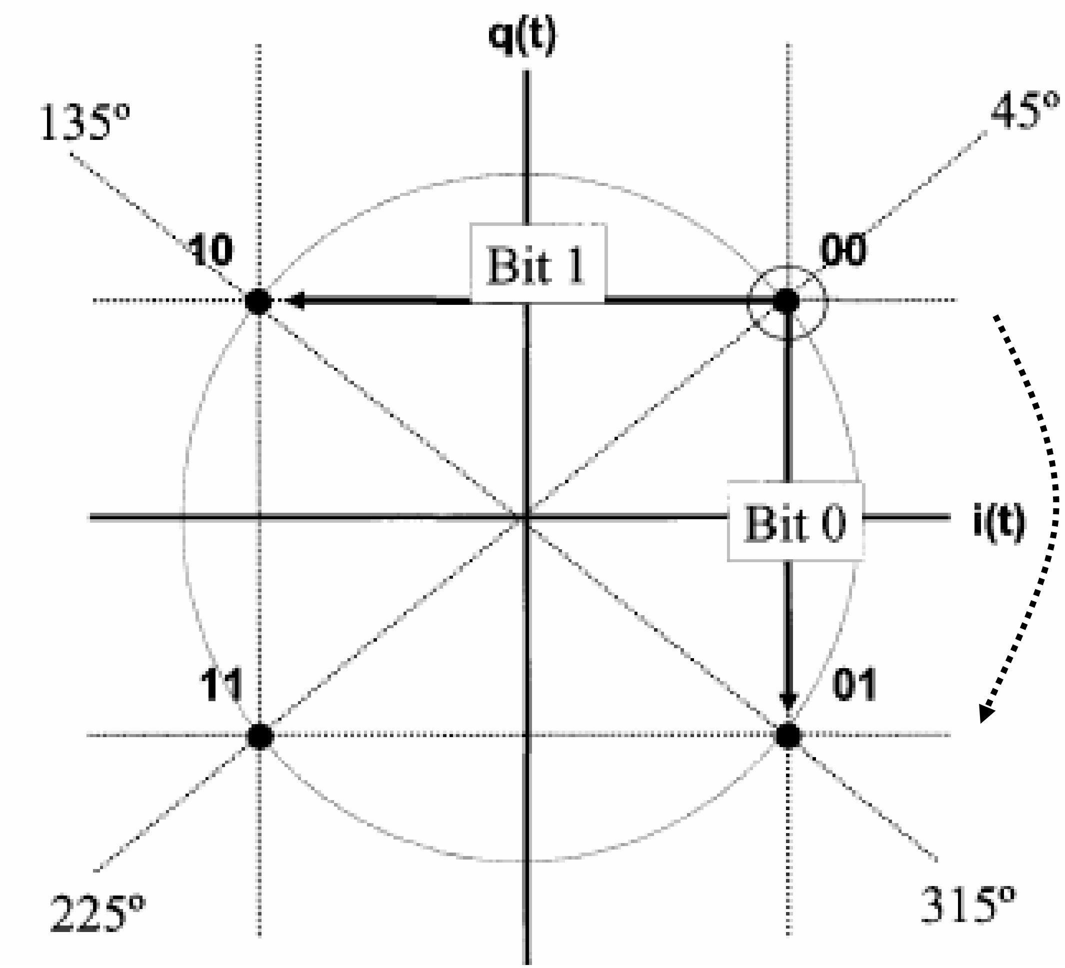
$$\boxed{P(e) \le \sum_{m=1}^{M} \sum_{\substack{k=1 \\ k \ne m}}^{M} P_{mk}^{e} P(\underline{s}_m) \le (M-1) Q\left(\frac{d_{\min}}{\sqrt{2N_o}} \right)}$$

(in general, a very poor estimate of the true error probability)

Quaternary Phase Shift Keying (QPSK)

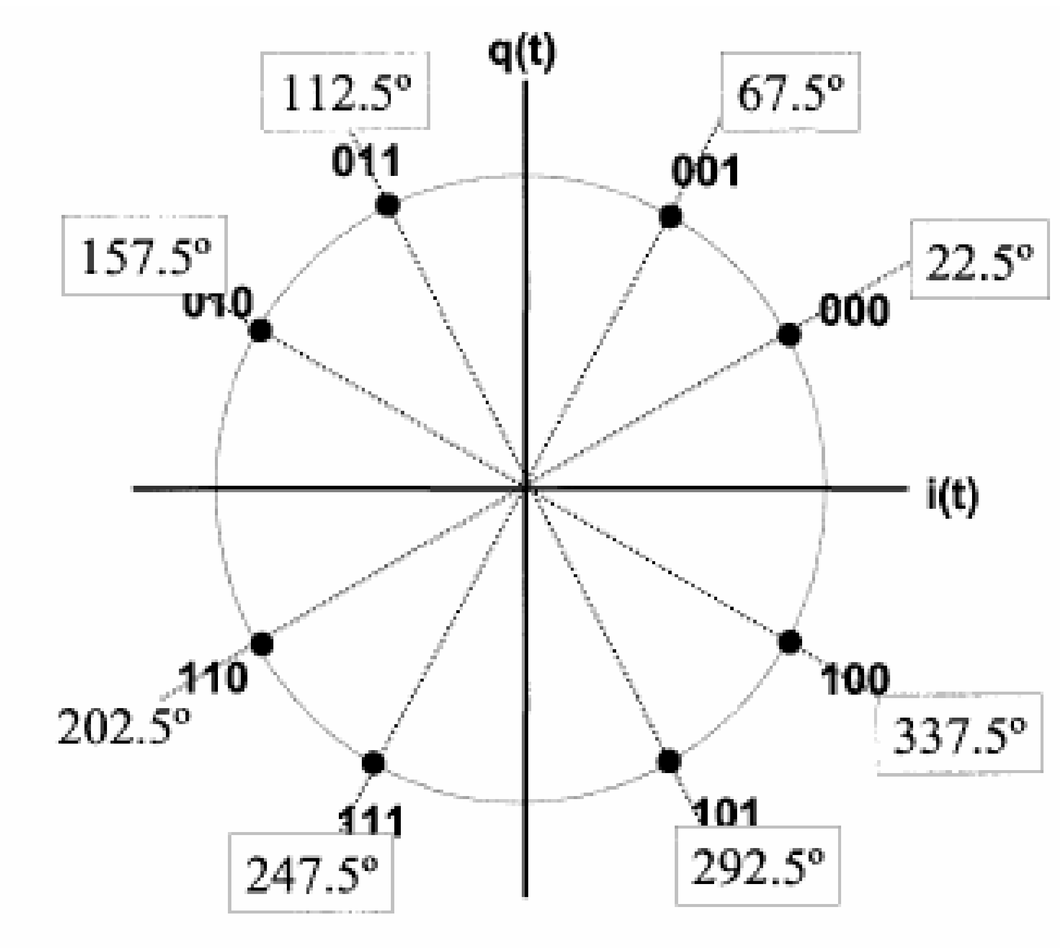


OQPSK/MSK/GMSK

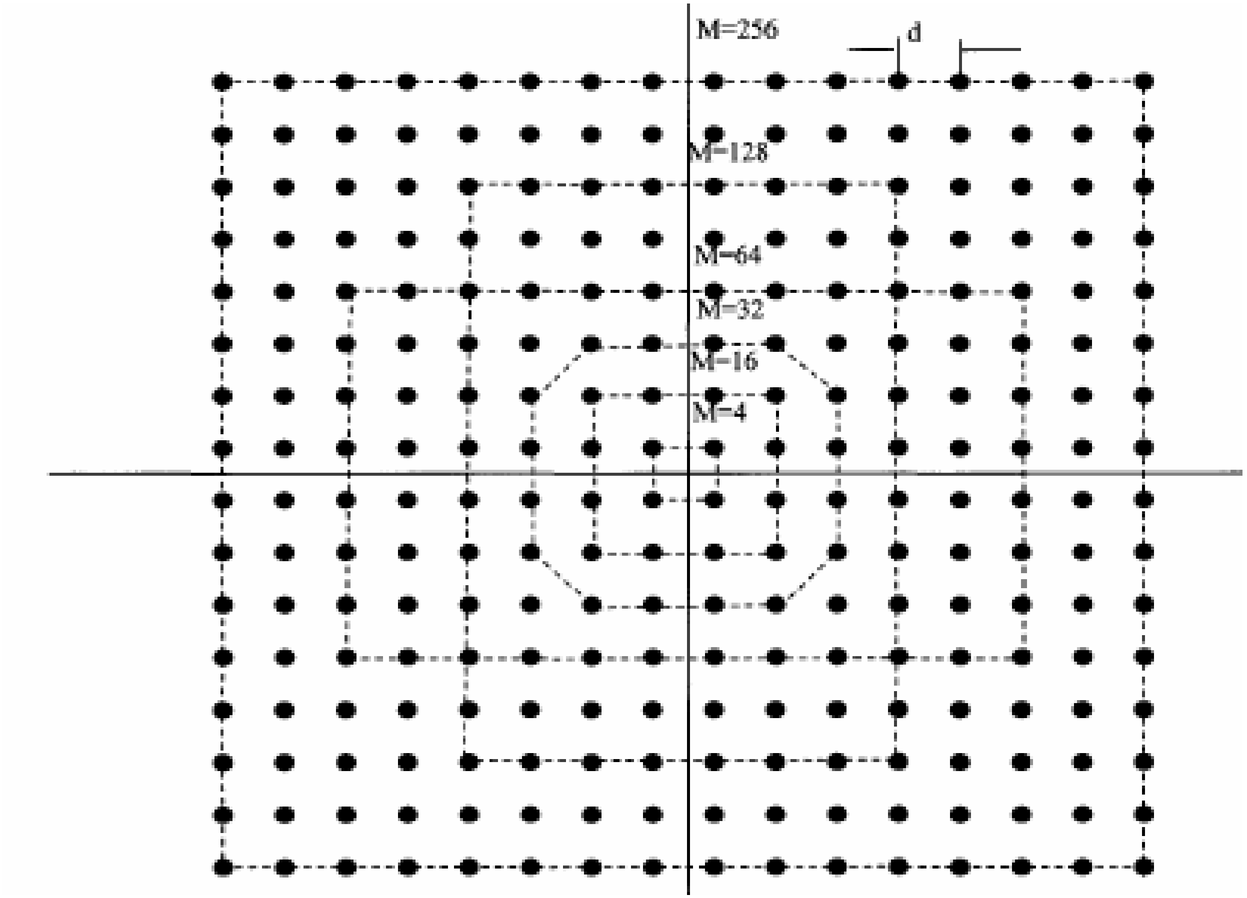


MSK/GMSK

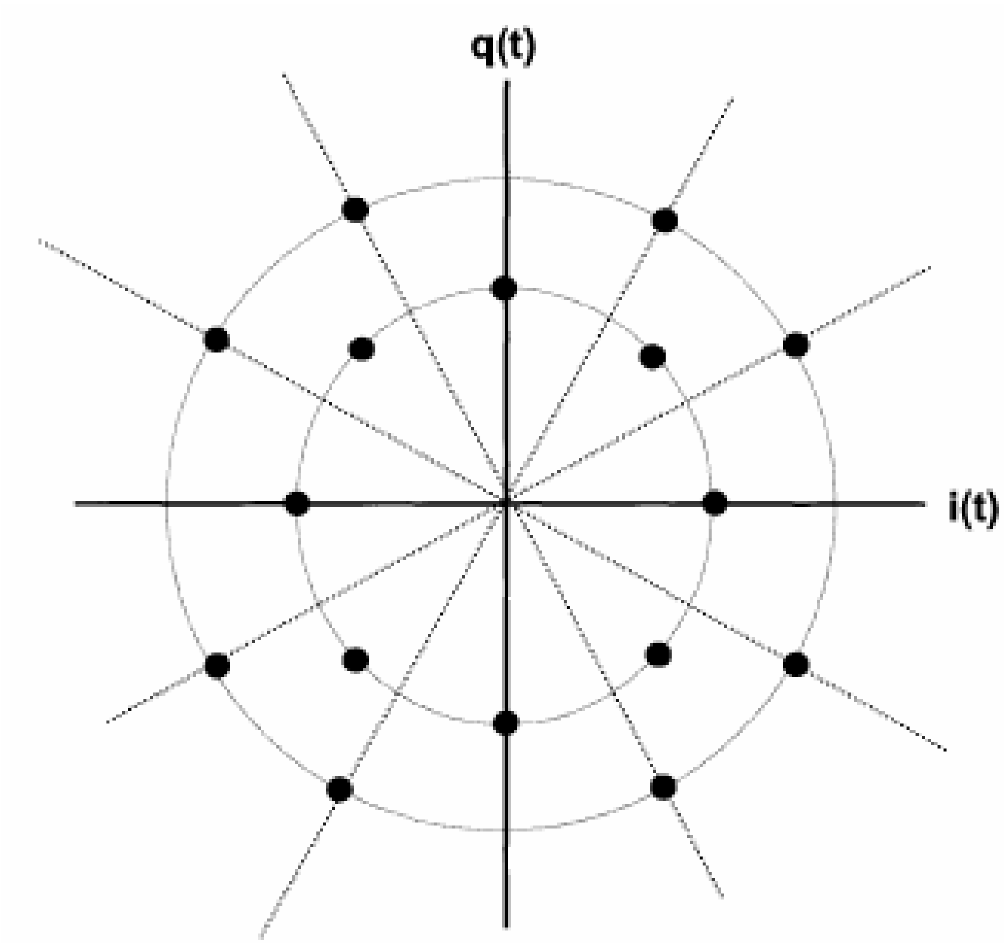
Octal PSK (OQPSK = 8PSK)



Quadrature Modulations (QAM)



APK Modulations



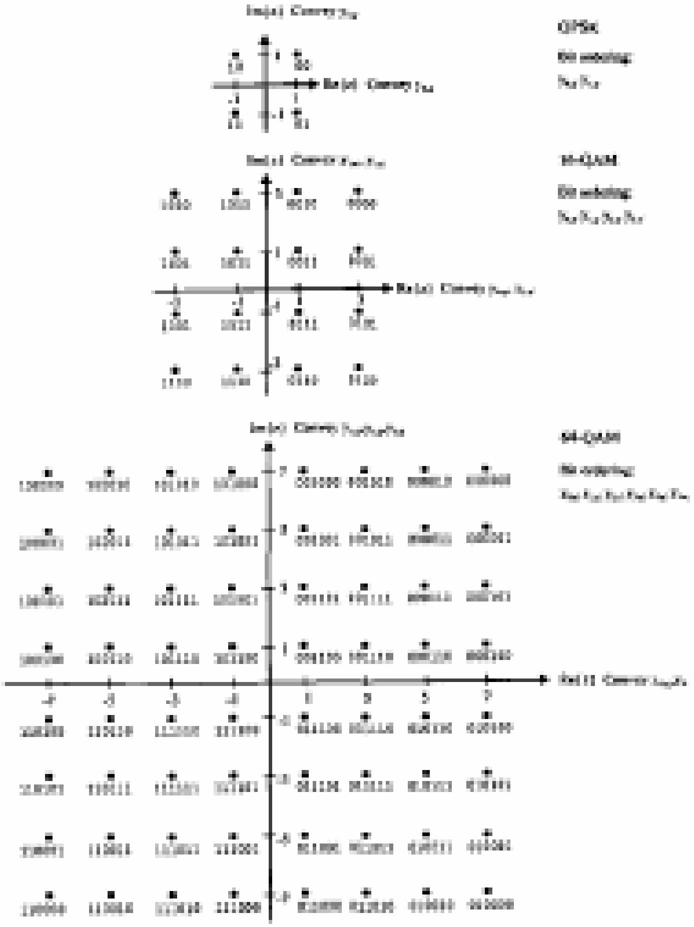


Figure 9a: The QPSK, 16-QAM and 64-QAM mappings and the corresponding bit patterns (non-hierarchical, and hierarchical with α = 1)

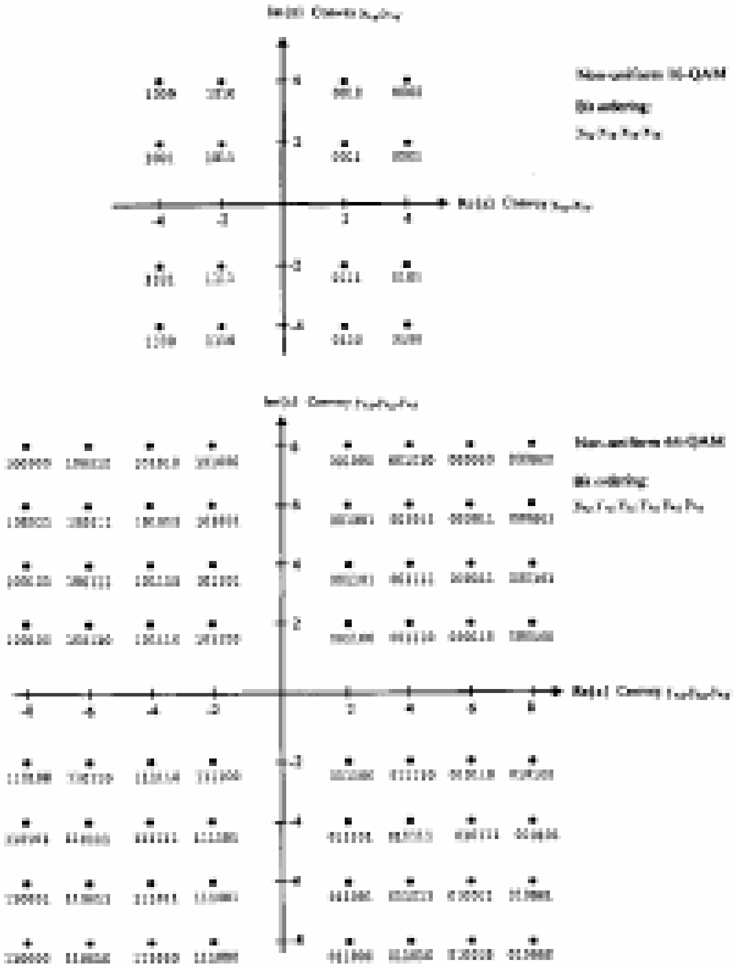


Figure 9b: Non-uniform 16-QAM and 64-QAM mappings with α = 2

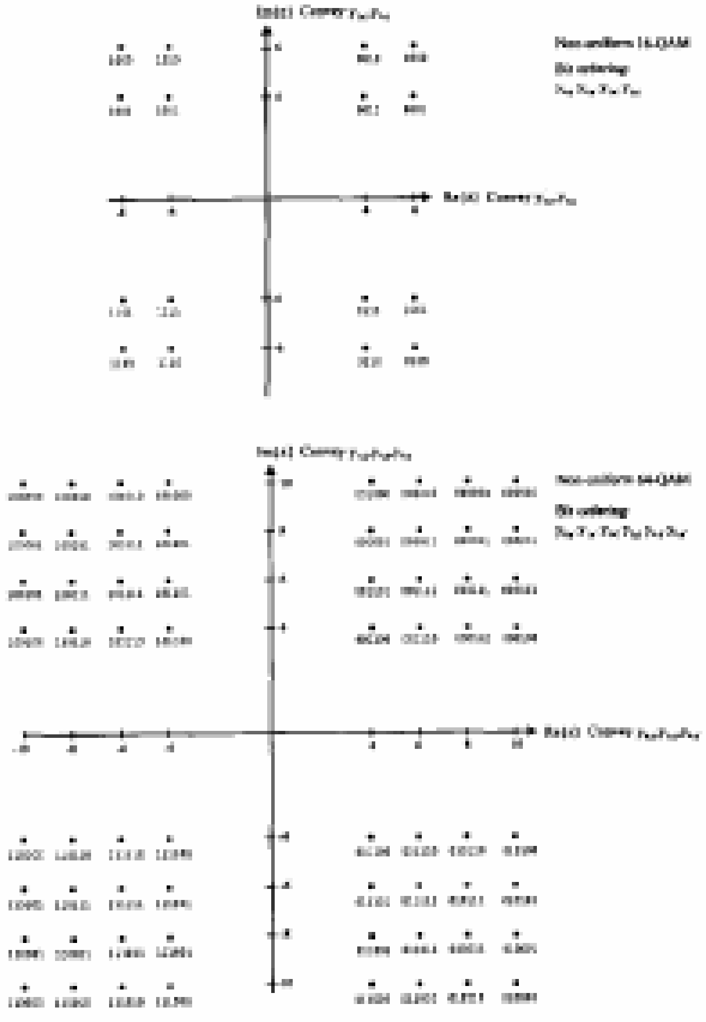


Figure 9b: Non-uniform 16-QAM and 64-QAM mappings with α = 4

THE BHATTACHARYYA-BOUND

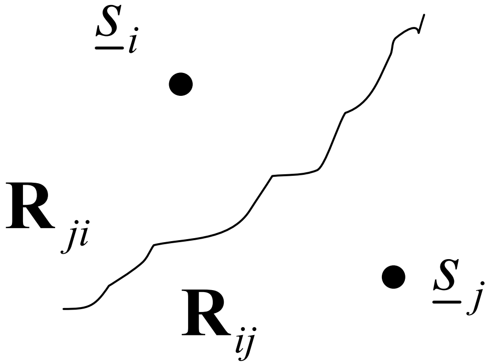
The pairwise error probability can be rewritten as follows:

$$P_e(i \to j) \equiv P_{ij}^e \equiv \Pr[\underline{v} \in \mathbf{R}_{ij} / \underline{s}_i] = \int_{\mathbf{R}_{ij}} f(\underline{v} / \underline{s}_i) d\underline{v}$$

where:

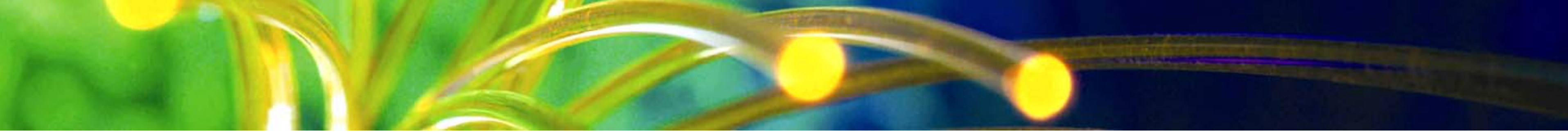
$$\mathbf{R}_{ij} \cup \mathbf{R}_{ji} = \mathbf{S}$$

$$\mathbf{R}_{ij} \equiv \left\{ \underline{v} \middle/ \underbrace{\frac{f(\underline{v} / \underline{s}_j)}{f(\underline{v} / \underline{s}_i)}}_{\text{Likelihood ratio}} \geq 1 \right\}$$



We can also define a function $q(\underline{v})$ **such that:**

$$q(\underline{v}) = \begin{cases} 1 & \underline{v} \in \mathbf{R}_{ij} \\ 0 & \underline{v} \notin \mathbf{R}_{ij} \end{cases} \qquad \text{such that:} \qquad P_e(i \to j) = P_{ij}^e = \int_{\mathbf{S}} q(\underline{v}) f(\underline{v} / \underline{s}_i) d\underline{v}$$



An upper-bound for q(v) is the following:

$$q(\underline{v}) = \begin{cases} 1 \le \sqrt{\dfrac{f(\underline{v}/\underline{s}_j)}{f(\underline{v}/\underline{s}_i)}} & \underline{v} \in \mathbf{R}_{ij} \\ \\ 0 \le \sqrt{\dfrac{f(\underline{v}/\underline{s}_j)}{f(\underline{v}/\underline{s}_i)}} & v \notin \mathbf{R}_{ij} \end{cases} \qquad \text{such that :}$$

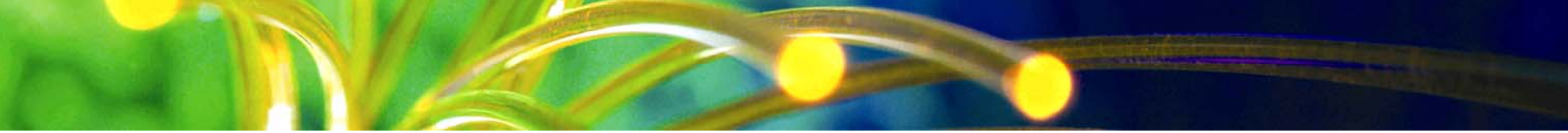
$$\boxed{P_e(i \to j) = \int_{\mathbf{S}} q(\underline{v}) f(\underline{v}/\underline{s}_i) d\underline{v} \le \int_{\mathbf{S}} \sqrt{f(\underline{v}/\underline{s}_i) f(\underline{v}/\underline{s}_j)}\, d\underline{v}}$$

Bhattacharyya-bound or distance

Thus, for equally likely symbols:

$$P(e) \le \sum_{i} \sum_{\substack{j \\ j \neq i}} P_e(i \to j) \le \sum_{i} \sum_{\substack{j \\ j \neq i}} \int_{\mathbf{S}} \sqrt{f(\underline{v}/\underline{s}_i) f(\underline{v}/\underline{s}_j)}\, d\underline{v}$$

union-bound

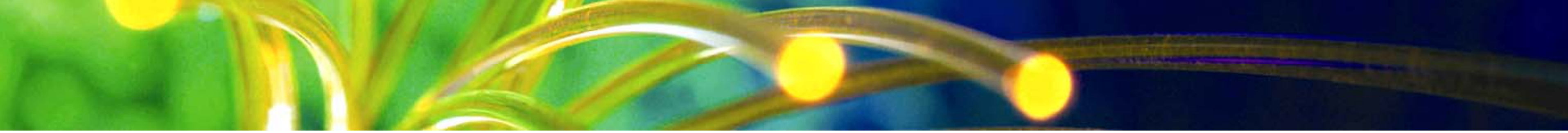


For the AWGN channel:

$$P_e(i \to j) \le \frac{1}{(\pi N_o)^{N/2}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{\left\|\underline{v}-\underline{s}_i\right\|^2}{2N_o}} e^{-\frac{\left\|\underline{v}-\underline{s}_j\right\|^2}{2N_o}} d\underline{v} = e^{-\frac{\left\|\underline{s}_i-\underline{s}_j\right\|^2}{4N_o}} = e^{-\frac{\left|d_{ij}\right|^2}{4N_o}}$$

Thus, the tightness of this bound is found from the comparison between:

$$\boxed{P_e(i \to j) = Q\left(\frac{\left\|\underline{s}_i - \underline{s}_j\right\|}{\sqrt{2N_o}}\right) = Q\left(\frac{d_{ij}}{\sqrt{2N_o}}\right)} \quad \text{vs} \quad \boxed{P_e(i \to j) \le e^{-\frac{\left|d_{ij}\right|^2}{4N_o}}}$$



THE GALLAGER-BOUND

We know that for equally likely symbols:

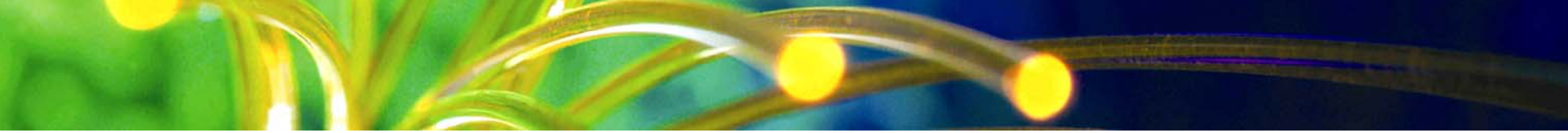
$$\mathbf{R}_i \equiv \left\{ \underline{v} \middle/ \frac{f(\underline{v}/\underline{s}_i)}{f(\underline{v}/\underline{s}_j)} \ge 1 \quad ;\mathrm{i} \ne \mathrm{j} \right\} \Rightarrow \overline{\mathbf{R}}_i \equiv \left\{ \underline{v} \middle/ \frac{f(\underline{v}/\underline{s}_j)}{f(\underline{v}/\underline{s}_i)} \ge 1 \quad ;\mathrm{i} \ne \mathrm{j} \right\}$$

Thus, if we focus on $\overline{\mathrm{R}}_{\mathrm{i}}$, we have that:

$$\frac{f(\underline{v}/\underline{s}_j)}{f(\underline{v}/\underline{s}_i)} \ge 1 \quad \mathrm{j} \ne i \quad \Rightarrow \quad \left[\frac{f(\underline{v}/\underline{s}_j)}{f(\underline{v}/\underline{s}_i)} \right]^{\lambda} \ge 1 \quad \text{for any } \lambda > 0$$

and then:

$$\sum_{\substack{j \\ j \ne i}} \left[\frac{f(\underline{v}/\underline{s}_j)}{f(\underline{v}/\underline{s}_i)} \right]^{\lambda} \ge 1 \qquad \forall \underline{v} \in \overline{\mathbf{R}}_i$$



We can now define the region $\tilde{\mathrm{R}}_\mathrm{i}$ such that:

$$\tilde{\mathbf{R}}_i \equiv \left\{ \underline{v} \middle/ \sum_{\substack{j \\ j\neq i}} \left[\frac{f(\underline{v}/\underline{s}_j)}{f(\underline{v}/\underline{s}_i)} \right]^{\lambda} \geq 1 \right\} \qquad \Rightarrow \qquad \overline{\mathbf{R}_i} \subset \tilde{\mathbf{R}}_i$$

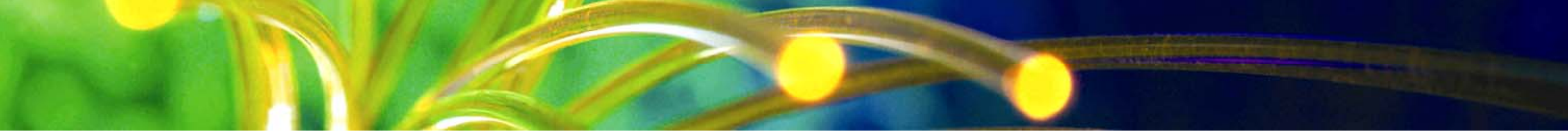
$$\Big\downarrow$$

and:

$$P(e/\underline{s}_i) = 1 - \int_{\mathbf{R}_i} f(\underline{v}/\underline{s}_i)\,d\underline{v} = \int_{\overline{\mathbf{R}_i}} f(\underline{v}/\underline{s}_i)\,d\underline{v} \leq \int_{\tilde{\mathbf{R}}_i} f(\underline{v}/\underline{s}_i)\,d\underline{v}$$

or, in other words:

$$P(e/\underline{s}_i) \leq \int_{\tilde{\mathbf{R}}_i} f(\underline{v}/\underline{s}_i)\,d\underline{v} = \int_{\mathbf{S}} q(\underline{v})\, f(\underline{v}/\underline{s}_i)\,d\underline{v} \qquad \text{for:} \quad q(\underline{v}) = \begin{cases} 1 & \text{for } \underline{v} \in \tilde{\mathbf{R}}_i \\ 0 & \text{for } \underline{v} \notin \tilde{\mathbf{R}}_i \end{cases}$$



Moreover:

$$q(\underline{v}) \leq \left[\sum_{\substack{j \\ j \neq i}} \left[\frac{f(\underline{v}/\underline{s}_j)}{f(\underline{v}/\underline{s}_i)} \right]^{\lambda} \right]^{\rho} \qquad \begin{array}{c} \forall \underline{v} \in \mathbf{S} \\ \lambda > 0 \\ \rho > 0 \end{array}$$

and finally:

$$P(e/\underline{s}_i) \leq \int_{\mathbf{S}} q(\underline{v}) f(\underline{v}/\underline{s}_i) d\underline{v} \leq \int_{\mathbf{S}} \left[f(\underline{v}/\underline{s}_i) \right]^{1-\lambda\rho} \left[\sum_{\substack{j \\ j \neq i}} (f(\underline{v}/\underline{s}_j))^{\lambda} \right]^{\rho} d\underline{v}$$

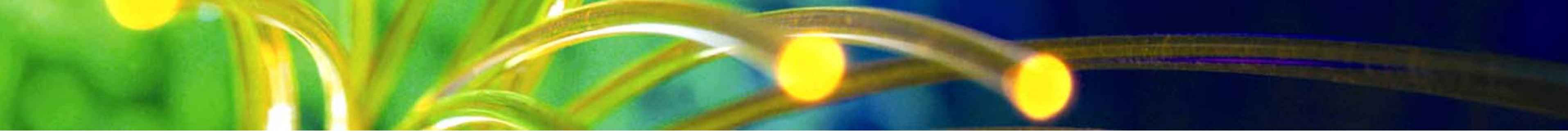
For:

$$\lambda = \frac{1}{1+\rho}$$

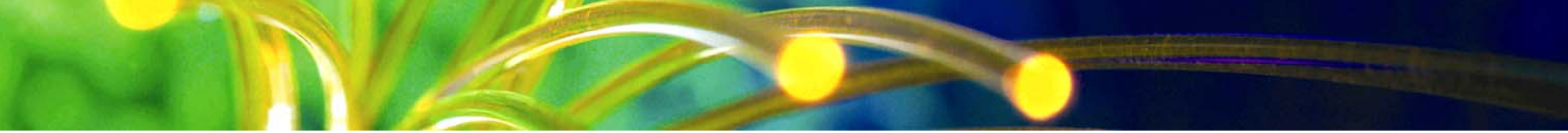
$$\boxed{P(e/\underline{s}_i) \leq \int_{\mathbf{S}} \left[f(\underline{v}/\underline{s}_i) \right]^{\frac{1}{1+\rho}} \left[\sum_{\substack{j \\ j \neq i}} (f(\underline{v}/\underline{s}_j))^{\frac{1}{1+\rho}} \right]^{\rho} d\underline{v}}$$

GALLAGER-BOUND

ρ=1 Bhattacharyya-bound



Example: ***ORTHOGONAL SIGNALLING***



For an alphabet A such that: $\underline{s}_m{}^T \underline{s}_n = \int_{-\infty}^{+\infty} s_m(t)s_n(t)dt = E\delta_{mn}$

we have that:

1.- All symbols have the same energy, that is: $E_m = \left\|\underline{s}_m\right\|^2 = E$

$\forall m = 1,2,\ldots, M$

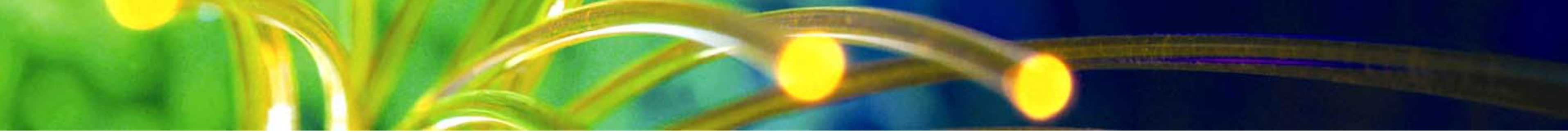
2.- The alphabet adopts an orthogonal signalling, or, in other words, N=M such that:

$$\underline{\phi}_m = \frac{1}{\sqrt{E}}\underline{s}_m \quad \text{(canonical representation)}$$

For an AWGN channel:

$$f(\underline{v}/\underline{s}_m) = f_{\underline{n}}(\underline{v}-\underline{s}_m) = \frac{1}{(\pi N_o)^{N/2}} e^{-\frac{\left\|\underline{v}-\underline{s}_m\right\|^2}{N_o}}$$

where: $\left\|\underline{v}-\underline{s}_m\right\|^2 = \left\|\underline{v}\right\|^2 - 2\underline{s}_m{}^T\underline{v} + \left\|\underline{s}_m\right\|^2 \Big\}\!\longrightarrow E_m = E \quad \forall m = 1, 2, \ldots, M = N$



Thus:

$$\left\|\underline{v}-\underline{s}_m\right\|^2=\left\|v\right\|^2-2\underline{s}_m{}^T\,\underline{v}+E$$

and:

m-th component of $\underline{v}$

$$\underline{s}_m{}^T\,\underline{v}=\int_{-\infty}^{+\infty}s_m(t)v(t)dt=\int_{-\infty}^{+\infty}\sqrt{E}\phi_m(t)v(t)dt=\sqrt{E}v_m$$

Then:

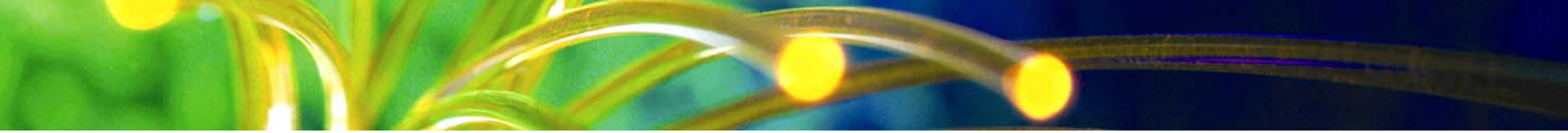
$$\left\|\underline{v}-\underline{s}_m\right\|^2=\left\|v\right\|^2-2\sqrt{E}v_m+E=\sum_{k=1}^{K}\left|v_k\right|^2-2\sqrt{E}v_m+E$$

And,

$$f(\underline{v}/\underline{s}_m)=\frac{1}{(\pi N_o)^{N/2}}e^{-\frac{\left\|\underline{v}-\underline{s}_m\right\|^2}{N_o}}=\frac{1}{(\pi N_o)^{N/2}}e^{-\frac{E-2\sqrt{E}v_m}{N_o}}e^{-\sum_{k=1}^{K}\frac{\left|v_k\right|^2}{N_o}}$$

From the Gallager's bound:

$$P(e/\underline{s}_m)\leq\int_{\mathbf{S}}\left[f(\underline{v}/\underline{s}_m)\right]^{\frac{1}{1+\rho}}\left[\sum_{\substack{m'\\ m'\neq m}}\left(f(\underline{v}/\underline{s}_{m'})\right)^{\frac{1}{1+\rho}}\right]^{\rho}d\underline{v}\qquad\qquad\rho\geq 0$$



Then, after some simple manipulations:

$$P(e/\underline{s}_m) \leq e^{\frac{-E}{No}} E[g(z_m)] E\left[\left(\sum_{\substack{m' \\ m' \neq m}} g(z_{m'})\right)^{\rho}\right] \qquad \rho \ \geq 0$$

where:

$$z_m \equiv \sqrt{\frac{2}{N_o}} v_m \quad \text{and} \quad g(z) \equiv e^{\sqrt{2\frac{E}{N_o}}\ \frac{z}{1+\rho}}$$

The evaluation of the first term:

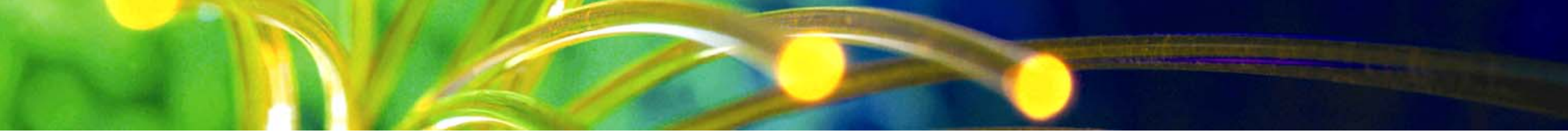
$$E[g(z)] = \int_{-\infty}^{+\infty} g(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = e^{\frac{E}{N_o (1+\rho)^2}}$$

The second term is not so trivial and it requires of the use of the *Jensen's inequality*:

Let's $f(x)$ a $\bigcap$ convex function, then :

$$E[f(x)] \leq f(E[x]) \longleftarrow$$

- Viterbi & Omura, pp.40 - 42
- Cover



If parameter ρ is constraint to $0 \leq \rho \leq 1$:

$$E\left[\left(\sum_{\substack{m' \\ m' \neq m}} g(z_{m'})\right)^{\rho}\right] \leq \left(E\left[\sum_{\substack{m' \\ m' \neq m}} g(z_{m'})\right]\right)^{\rho} = (M-1)^{\rho}\left(E\left[g(z)\right]\right)^{\rho}$$

and:

$$P\left(e/\underline{s}_m\right) \leq (M-1)^{\rho}\, e^{-\frac{E}{N_o}}\left(E\left[g(z)\right]\right)^{\rho+1}$$

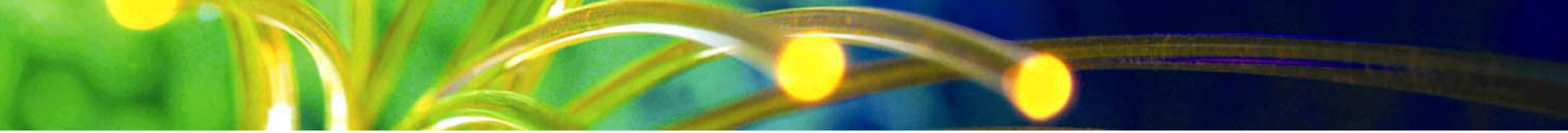
Thus:

$$P(e) = \sum_{m=1}^{M} P\left(e/\underline{s}_m\right) P\left(\underline{s}_m\right) = P\left(e/\underline{s}_m\right) \underbrace{\sum_{m=1}^{M} P\left(\underline{s}_m\right)}_{1} = P\left(e/\underline{s}_m\right)$$

or:

$$\boxed{P(e) \leq (M-1)^{\rho}\, e^{-\frac{E}{N_o}\left(\frac{\rho}{1+\rho}\right)} \qquad 0 \leq \rho \leq 1}$$

It is now necessary to optimize P(e) on 'ρ '



Some definitions:

1. $r = \dfrac{1}{T} \quad signalling - rate\ (symbols/\sec).$

2. $\gamma = \left(\dfrac{E}{N_o}\right)\dfrac{1}{T} = \dfrac{S}{N_o} \quad signal\ to\ noise - density\ ratio\ ('Hz')\ ../...$

3. $bit - rate = r_b = \dfrac{\ln M}{T}\ (nats/\sec)$

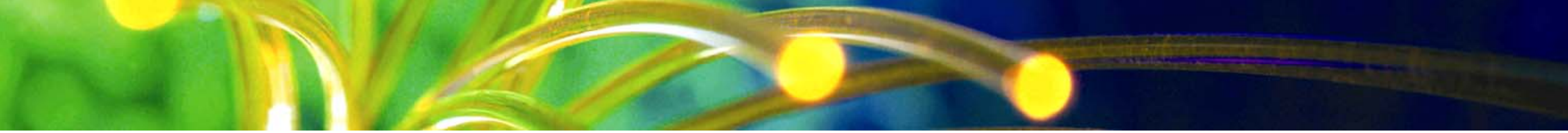
The optimization of P(e) on ‘ ρ ‘ implies that:

$$\frac{d}{d\rho}P(e) = 0 \quad \Rightarrow \quad \frac{d}{d\rho}\ln P(e) = 0 \quad \Rightarrow \quad (1+\rho)^2 = \frac{\gamma}{r_b} \quad \Rightarrow \quad \begin{cases} \rho = \sqrt{\dfrac{\gamma}{r_b}} - 1 \\ 0 \le \rho \le 1 \end{cases}$$

that is:

$$1 \le \sqrt{\frac{\gamma}{r_b}} \le 2 \quad \Rightarrow \quad 1 \le \frac{\gamma}{r_b} \le 4$$

$$\frac{1}{2} \le \sqrt{\frac{r_b}{\gamma}} \le 1 \quad \Rightarrow \quad \frac{1}{4} \le \frac{r_b}{\gamma} \le 1$$



Importance of ‘ γ ‘:

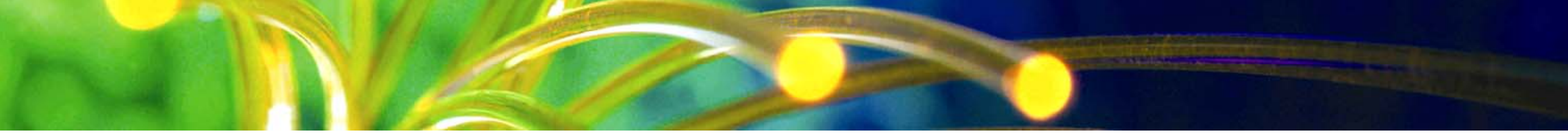
$$\gamma = \left(\frac{E}{N_o}\right)\frac{1}{T} = \frac{S}{N_o} \quad \begin{array}{l} \rightarrow RX\ signal\ power \\ \rightarrow quality\ of\ the\ RX \end{array}$$

reception ‘conditions’

Thus:

$$\boxed{\frac{\gamma}{r_b} = \frac{1}{\ln M}\left(\frac{E}{N_o}\right) = \frac{E_b}{N_o}\log_2 e}$$

traffic requirement



What about $\frac{r_b}{\gamma} < \frac{1}{4}$ **?:**

for such case $\rho > 1$ **!! and then, the optimal value becomes** $\rho = 1$

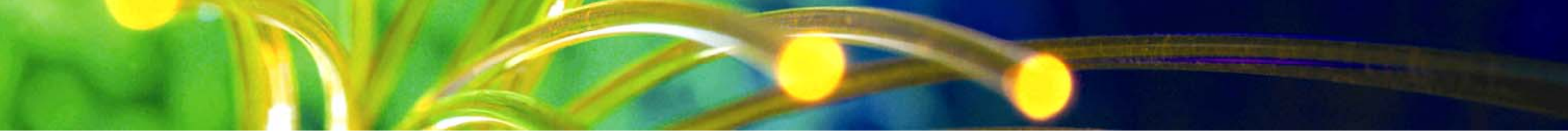
Thus, we will consider two intervals:

$$\left|\begin{array}{lll} (A): & \frac{1}{4} \le \frac{r_b}{\gamma} \le 1 & \textit{corresponding to } 0 \le \rho \le 1 \\ (B): & \frac{r_b}{\gamma} < \frac{1}{4} & \textit{corresponding to } \rho = 1 \, \text{(upper limit of the analysis)} \end{array}\right.$$

The last step is to find a function E[r_b] such that:

$$\boxed{P(e) < e^{-TE[r_b]}}$$

where E[r_b] is the so-called '*Reliability Function*'



For doing so, we have that:

$$P\left(e\right)\le\left(M-1\right)^{\rho}e^{-\frac{E}{N_o}\left(\frac{\rho}{1+\rho}\right)}<M^{\rho}e^{-\frac{E}{N_o}\left(\frac{\rho}{1+\rho}\right)}\underbrace{=}_{T\gamma=\frac{E}{N_o}}e^{\rho\ln M}e^{-T\gamma\frac{\rho}{1+\rho}}=$$

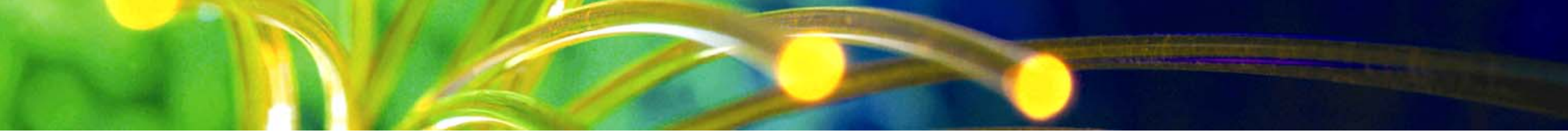
$$\underbrace{=}_{r_b=\frac{\ln M}{T}}e^{\rho T r_b}e^{-T\gamma\frac{\rho}{1+\rho}}=\boxed{e^{-T\left[\gamma\frac{\rho}{1+\rho}-\rho r_b\right]}\equiv e^{-TE\left[r_b\right]}}$$

For interval (A), that is, $\rho_{opt}=\sqrt{\frac{\gamma}{r_b}}-1$

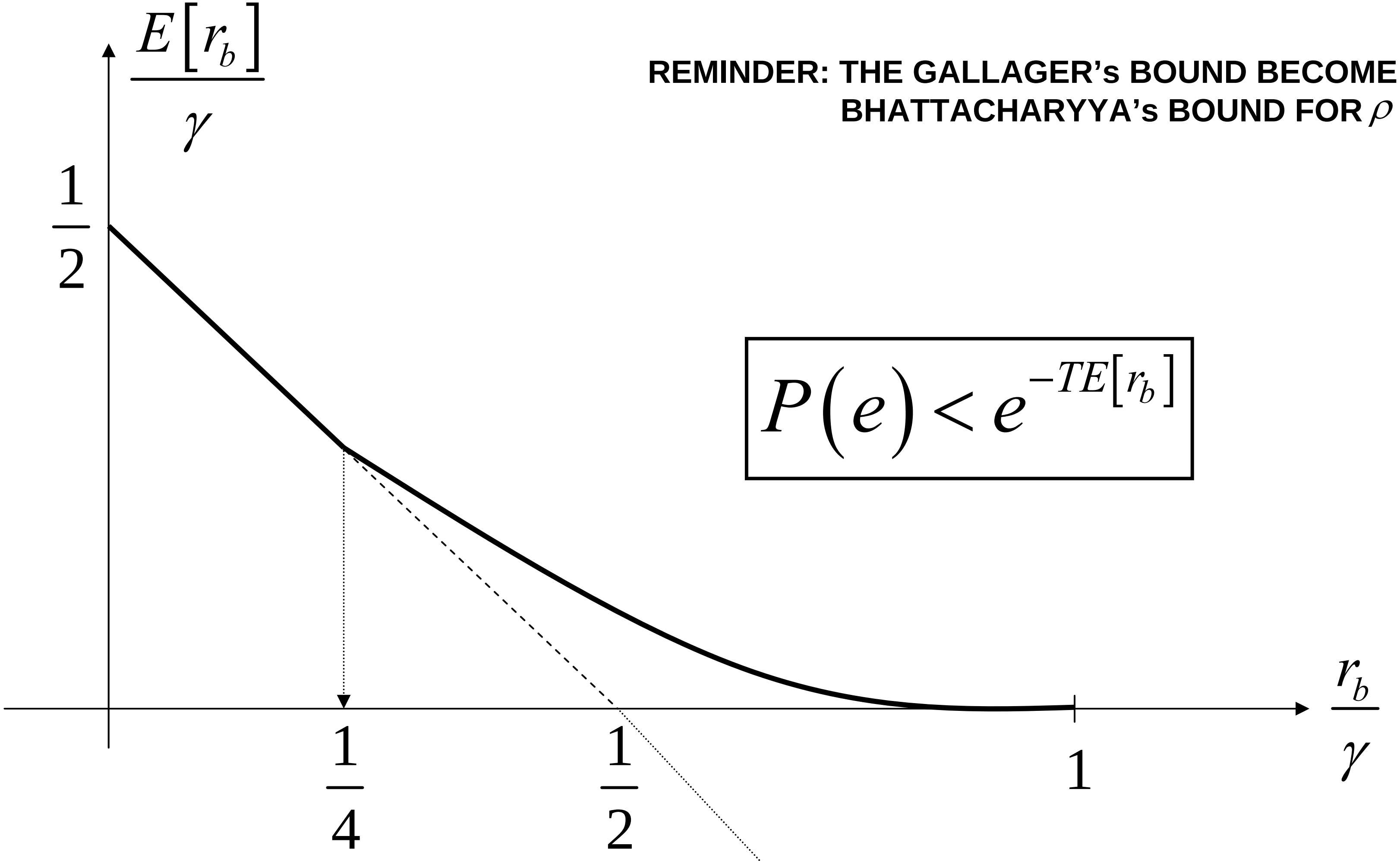
$$\boxed{E\left[r_b\right]=\left(\sqrt{\gamma}-\sqrt{r_b}\right)^2\quad\frac{1}{4}\le\frac{r_b}{\gamma}\le1}$$

For interval (B), that is, $\rho_{opt}=1$

$$\boxed{E\left[r_b\right]=\frac{1}{2}\gamma-r_b\quad 0\le\frac{r_b}{\gamma}\le\frac{1}{4}}$$

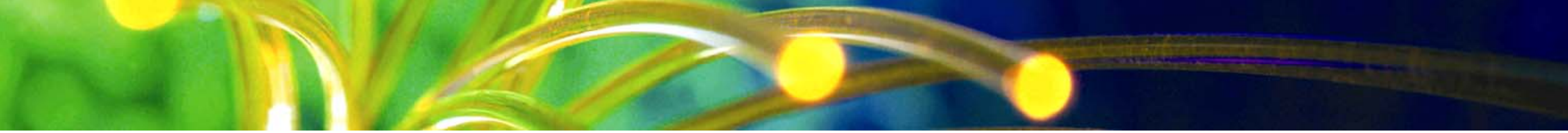


REMINDER: THE GALLAGER's BOUND BECOME THE BHATTACHARYYA's BOUND FOR $\rho = 1$



$$\boxed{P(e) < e^{-TE[r_b]}}$$

BHATTACHARYYA's PREDICTION



Finally:

$$r_b = \frac{\ln M}{T}\, nats/\sec$$

for r$_b$=ctn. (traffic requirement):

$$T \to \infty \quad \Rightarrow \quad M \to \infty$$

and:

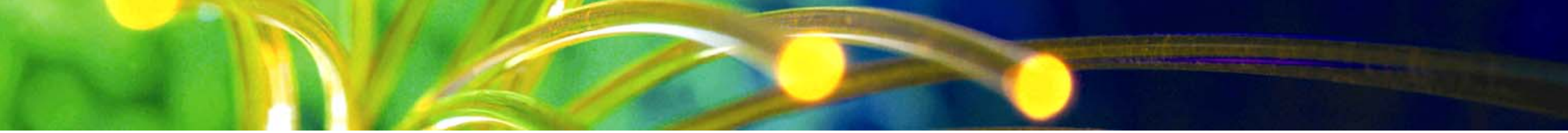
$$\lim_{T\to\infty} P(e) = \lim_{T\to\infty} e^{-TE[r_b]} \to 0!! \quad for \quad \frac{r_b}{\gamma} < 1$$

or, in other words:

$$r_b < \gamma = \frac{S}{N_o}\, nats/\sec$$

that is, once more:

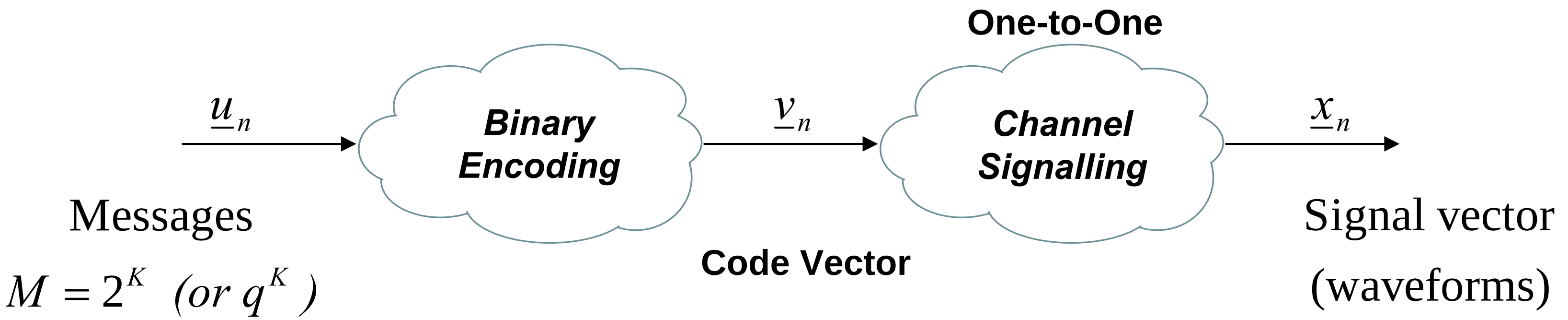
$$\boxed{\frac{r_b}{\gamma} = \frac{N_o}{E_b} < \frac{1}{\log_2 e} \approx -1.6\, dB \quad for \quad r_b\ bits/\sec}$$



Example:

PARITY-CHECK CODES or GROUP CODES (LINEAR CODES)

EXAMPLE: *Parity-Check Codes/Group Codes* ← *Linear Codes*



$$\underline{\mathrm{u}}_{\mathrm{n}} = [\mathrm{u}_{\mathrm{n}_1}, \mathrm{u}_{\mathrm{n}_2}, \ldots, \mathrm{u}_{\mathrm{n}_K}] \qquad \underline{\mathrm{v}}_{\mathrm{n}} = [\mathrm{v}_{\mathrm{n}_1}, \mathrm{v}_{\mathrm{n}_2}, \ldots, \mathrm{v}_{\mathrm{n}_L}] \qquad \underline{\mathrm{x}}_{\mathrm{n}} = [\mathrm{x}_{\mathrm{n}_1}, \mathrm{x}_{\mathrm{n}_2}, \ldots, \mathrm{x}_{\mathrm{n}_N}]$$

$$\boxed{\underline{\mathrm{v}}_{\mathrm{n}} = \underline{\mathrm{u}}_{\mathrm{n}} \underline{\underline{G}}}$$ **linear structure**

We want to *characterize the performance improvement* of such schemes!

Define the modulo-2 addition of binary symbols

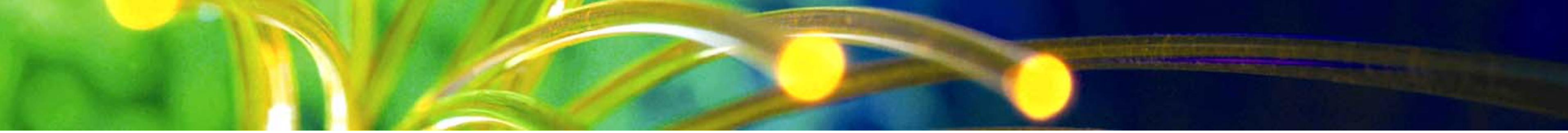
$$0 \oplus 1 \equiv 1 \oplus 0 \equiv 1$$

$$0 \oplus 0 \equiv 1 \oplus 1 \equiv 0$$

We see that:

1.–	$a \oplus b \equiv b \oplus a$	Commutative
2.–	$(a \oplus b) \oplus c \equiv a \oplus (b \oplus c)$	Associative
3.–	$a \oplus 0 = a$	Identity
4.–	$a \oplus a = 0$	Additive inverse or negative

→ Vectorial formulation



Define the modulo-2 product of binary symbols

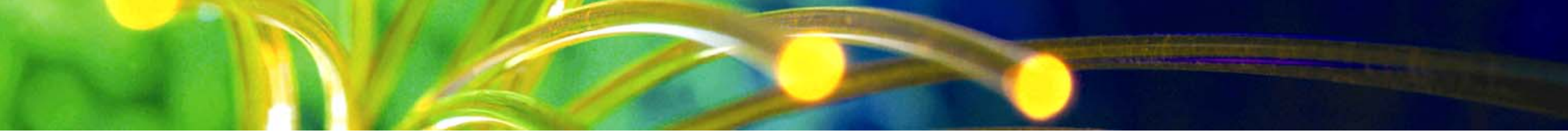
$$0x1 \equiv 1x0 \equiv 0x0 \equiv 0$$

$$1x1 \equiv 1$$

We see that:

$$1.- \quad 1xa \equiv a$$

$$2.- \quad ax(b \oplus c) = axb \oplus axc$$



If one considers linear encoding schemes (<u>parity check</u>)

$$\left.\begin{array}{l} v_{n_1} = u_{n_1} g_{11} \oplus u_{n_2} g_{21} \oplus \cdots \oplus u_{n_K} g_{K1} \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ v_{n_L} = u_{n_1} g_{1L} \oplus u_{n_2} g_{2L} \oplus \cdots \oplus u_{n_K} g_{KL} \end{array}\right\}$$

We have that:

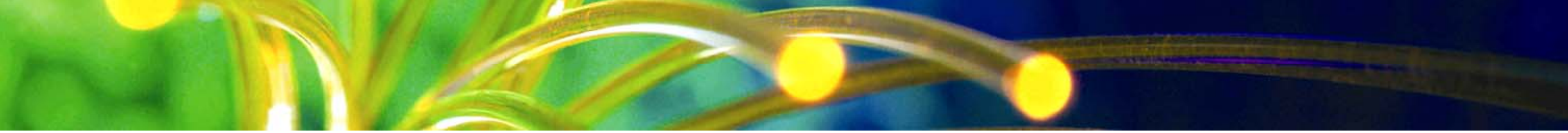
$$\underline{v}_n = \underline{u}_n \underline{\underline{G}} = \sum_{k=1}^{K} u_{n_k} \underline{g}_k$$

where:

$$\underline{\underline{G}} = \underset{\xleftrightarrow[\text{Generator matrix}]{}}{\begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1L} \\ \vdots & & & \vdots \\ \longleftarrow & \underline{g}_k & & \longrightarrow \\ \vdots & & & \vdots \\ g_{K1} & g_{K2} & \cdots & g_{KL} \end{bmatrix}}$$

$$\underline{\mathrm{u}}_{\mathrm{n}} = [\mathrm{u}_{\mathrm{n}_1}, \mathrm{u}_{\mathrm{n}_2}, \ldots, \mathrm{u}_{\mathrm{n}_\mathrm{K}}]$$

$$\underline{\mathrm{g}}_{\mathrm{k}} \equiv [\mathrm{g}_{\mathrm{k}_1}, \mathrm{g}_{\mathrm{k}_2}, \ldots, \mathrm{g}_{\mathrm{k}_\mathrm{L}}]$$



That is:

1.- “*Closure*” property: the sum of two code vectors is a code vector

$$\underline{v}_m \oplus \underline{v}_n = \underline{u}_m \underline{\underline{G}} \oplus \underline{u}_n \underline{\underline{G}} = (\underline{u}_m \oplus \underline{u}_n)\underline{\underline{G}}$$

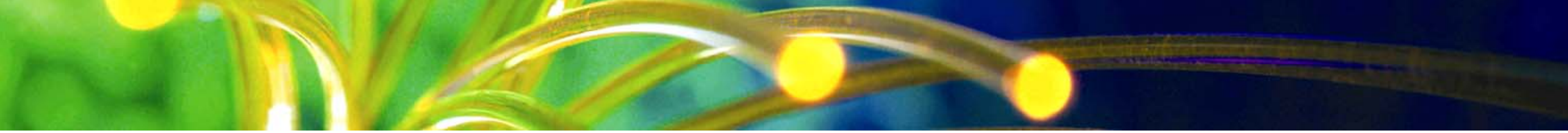
2.- Identity vector:

$$\underline{v}_m \oplus \underline{0} = \underline{v}_m$$

3.- Additive inverse:

$$\underline{v}_m \oplus \underline{v}_m = 0$$

4.- Commutative and associative.



***HAMMING WEIGHT*: Number of ones in the vector, that is,**

$$\omega(\underline{v}_n) \equiv \underline{v}_n \underline{\mathbf{1}}^T \qquad \text{(regular inner product)}$$

for: $\underline{\mathbf{1}} = [1\ 1\ 1\ \cdots\ 1]$

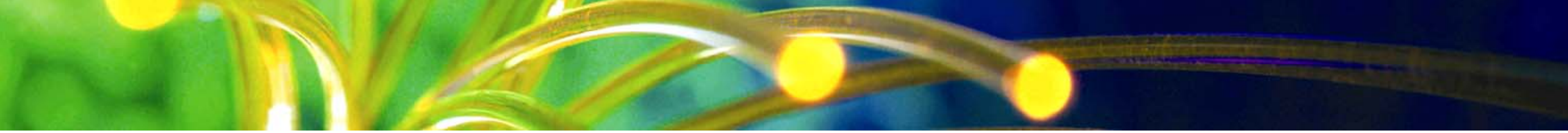
***HAMMING DISTANCE*:**

$$\boxed{\mathrm{d}_{\mathrm{H}}(\underline{v}_m;\underline{v}_n) \equiv \omega(\underline{v}_m \oplus \underline{v}_n)}$$

We define also the set of distances from a code $\underline{v}_m$ to the other codes:

$$D_m \equiv \left\{\omega(\underline{v}_m \oplus \underline{v}_n) \big/ \forall m \neq n\right\} = D$$

This set does not depend on ‘m’



To validate the last statement, we can see that:

1.- For the identity vector $\underline{v}_1 = \underline{0} \rightarrow \underline{v}_1 \oplus \underline{v}_m = \underline{v}_m$

$$D_1 \equiv \left\{ \omega(\underline{0} \oplus \underline{v}_m) \quad \forall m \neq 1 \right\} = \left\{ \omega(\underline{v}_2), \omega(\underline{v}_3), \ldots, \omega(\underline{v}_m) \right\}$$

2.- For $m \neq n,\ n \neq p,\ m \neq p:$

$$\underline{v}_n \oplus \underline{v}_m \neq \underline{v}_p \oplus \underline{v}_m$$

$$\underline{v}_n \oplus \underline{v}_m \neq \underline{0} = \underline{v}_1$$

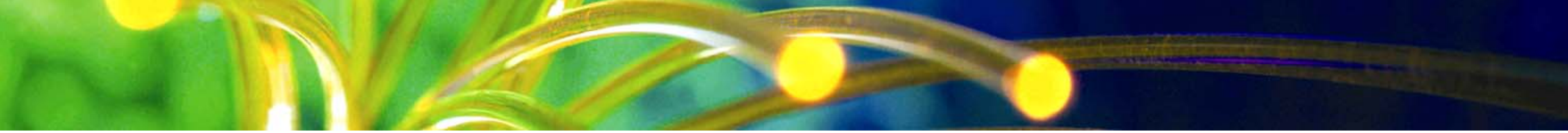
Additive inverse is unique

3.- $\left\{ \underline{v}_n \oplus \underline{v}_m; \quad \forall \mathrm{n} \neq \mathrm{m} \right\} = \left\{ \underline{v}_2, \underline{v}_3, \ldots, \underline{v}_M \right\}$

$$\underline{v}_n = \underline{v}_n \oplus \underline{0} = \underline{v}_n \oplus (\underline{v}_k \oplus \underline{v}_k) = (\underline{v}_n \oplus \underline{v}_k) \oplus \underline{v}_k$$

4.- Thus: $D_m \equiv \left\{ \omega(\underline{v}_n \oplus \underline{v}_m); \quad \forall n \neq m \right\} = \left\{ \omega(\underline{v}_2), \omega(\underline{v}_3), \ldots, \omega(\underline{v}_M) \right\}$

$$\forall m \qquad \boxed{D_m = D}$$



Binary signalling/mapping:

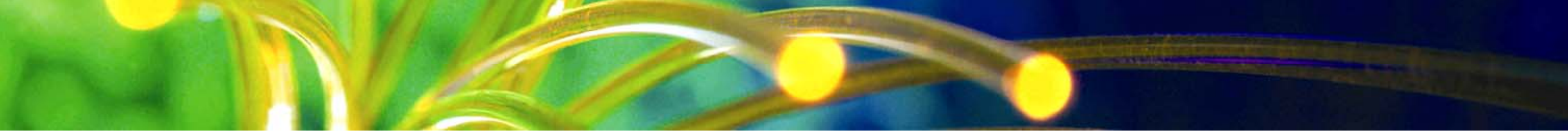
$$\underline{v}_n \qquad \left.\begin{array}{l} v_{np} = 0 \quad \text{to} \quad x_{np} = +\sqrt{E} \\ v_{np} = 1 \quad \text{to} \quad x_{np} = -\sqrt{E} \end{array}\right\} \text{Equally likely}$$

Or the equivalent signal-space notation:

$\underline{x}_n$ can adopt M different waveforms $\mathbf{A} = \left\{\underline{s}_m(t)\right\}_{1 \le m \le M}$

$$\underline{x}_n = [x_{n_1}, x_{n_2}, \ldots, x_{n_N}] \;\Rightarrow\; \textbf{M=2}^{\textbf{N}} \textbf{ symbols or waveforms}$$

and use the equivalence between $\underline{x}_n$ or $\underline{s}_n$



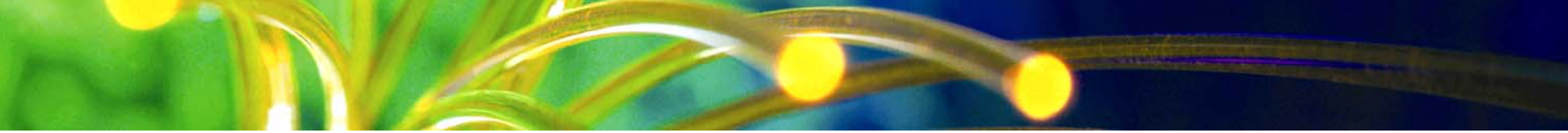
***Lemma*:** **For a <u>input-binary</u>, <u>output-symmetric</u> channel, a maximum likelihood decoder will supply an error probability equal for all the symbols of the alphabet, that is:**

$$P(e/\underline{s}_m) = ctn \qquad \forall m = 1,2,\ldots,M$$

and thus:

$$P(e) = \sum_{m=1}^{M} P(e/\underline{s}_m)P(\underline{s}_m) = P(e/\underline{s}_m) \qquad m = 1,2,\ldots,M$$

***Consequence*:** **The performance characterization of the code can be done for any symbol of the constellation or code vector (code word).**



The computation of the conditional probability given by:

$$P(e/\underline{x}_n) \quad \text{or} \quad P(e/\underline{s}_n)$$

requires the identification of $\overline{\mathrm{R}}_n$, that is,

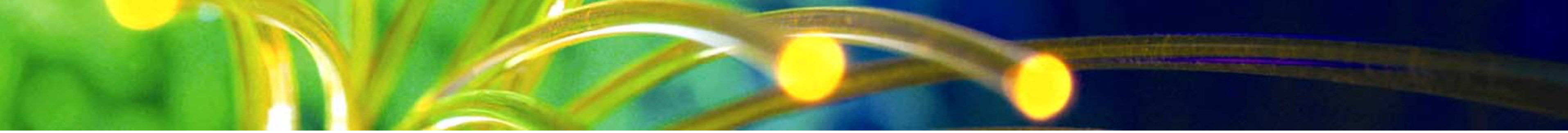
$$P(e/\underline{x}_n) = \Pr\left\{\underline{v} \in \overline{\mathbf{R}_n} / \underline{x}_n\right\} \;\; = \int_{\overline{\mathbf{R}}_n} f(\underline{v}/\underline{s}_n)d\underline{v}$$

where, for a ML decoder:

$$\overline{\mathbf{R}_n} \equiv \left\{\underline{v}/\ln f(\underline{v}/\underline{s}_m) \geq \ln f(\underline{v}/\underline{s}_n); \quad \mathrm{m} \neq \mathrm{n}\right\}$$

For any component of the observation:

$$\left.\begin{array}{l} f(v_k / x_{n_k} = +\sqrt{E}) = f(v_k / v_{n_k} = 0) \equiv f_0(v_k) \\ f(v_k / x_{n_k} = -\sqrt{E}) = f(v_k / v_{n_k} = 1) \equiv f_1(v_k) \end{array}\right\}$$



The symmetry of the channel implies that:

$$\boxed{f_1(v_k) = f_o(-v_k) \qquad \forall k}$$

Thus:

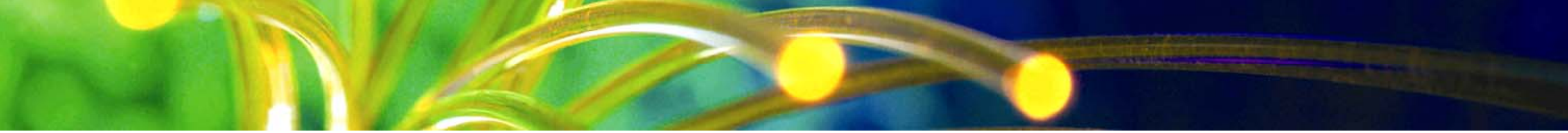
$$P(e/\underline{x}_n) = P(e/\underline{s}_n) = \int_{\overline{\mathbf{R}_n}} f(\underline{v}/\underline{x}_n)d\underline{v} =$$

$$= \int_{\overline{\mathbf{R}_n}} \prod_{\substack{k \\ x_{n_k}=+\sqrt{E}}} \underbrace{f(v_k / x_{n_k} = +\sqrt{E})}_{f_0(v_k)} \prod_{\substack{k \\ x_{n_k}=-\sqrt{E}}} \underbrace{f(v_k / x_{n_k} = -\sqrt{E})}_{f_1(v_k)=f_0(-v_k)} d\underline{v}$$

Thus, the integration will be performed such that:

$$\text{for} \quad x_{n_k} = +\sqrt{E} \qquad\qquad \mu_k \equiv v_k$$

$$\text{for} \quad x_{n_k} = -\sqrt{E} \qquad\qquad \mu_k \equiv -v_k$$



or, in other words,

$$P(e/\underline{s}_n) = P(e/\underline{x}_n) = \int_{\overline{\mathbf{R}}_n} \underbrace{\prod_{k=1}^{K} f_0(v_k)\,d\underline{v}}_{\text{Independent on 'n'}}$$

What about $\overline{R}_n$?. The procedure is the same, that is:

$$f(\underline{v}/\underline{x}_n) = \prod_{\substack{k \\ x_{n_k}=+\sqrt{E}}} f(v_k / x_{n_k} = +\sqrt{E}) \prod_{\substack{k \\ x_{n_k}=-\sqrt{E}}} f(v_k / x_{n_k} = -\sqrt{E}) =$$

$$= \prod_{\substack{k \\ x_{n_k}=+\sqrt{E}}} f_0(v_k) \prod_{\substack{k \\ x_{n_k}=-\sqrt{E}}} f_0(-v_k)$$

and similarly for:

$$f(\underline{v}/\underline{x}_m)$$

And, then, the likelihood ratio becomes:

$$\boxed{\ln\frac{f(\underline{v}/\underline{x}_m)}{f(\underline{v}/\underline{x}_n)} = \sum_{\substack{k \\ x_{n_k} \neq x_{m_k}}} \ln\frac{f_0(-v_k)}{f_0(v_k)} \geq 0 \qquad \text{also independent on 'n' !!}}$$

Notice the integration dummy variables!

Thus:

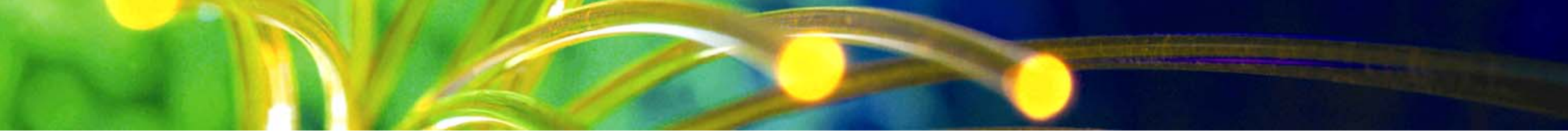
$$P(e) = P(e/\underline{s}_n) = P(e/\underline{x}_n) \qquad \forall n = 1,2,\ldots,M$$

and we can consider any code in the computation of P(e).

We will adopt the *all-zero* code, because:

1.- Due to the linearity of the code, it always belongs to the codebook

2.- It simplifies the analysis



$$P\left(e/\underline{\mathbf{s}}_m\right)=\int_{\bar{\mathbf{R}}_m}\prod_{\substack{k\\ x_{mk=+\sqrt{E}}}} f_o\left(\upsilon_k\right)\prod_{\substack{k\\ x_{mk=-\sqrt{E}}}} f_o\left(-\upsilon_k\right)d\boldsymbol{\upsilon}=\sum_{\substack{n=1\\ n\neq m}}^{M}\int_{\mathbf{R}_n}\prod_{\substack{k\\ x_{mk=+\sqrt{E}}}} f_o\left(\upsilon_k\right)\underline{\prod_{\substack{k\\ x_{mk=-\sqrt{E}}}} f_o\left(-\upsilon_k\right)}d\boldsymbol{\upsilon}$$

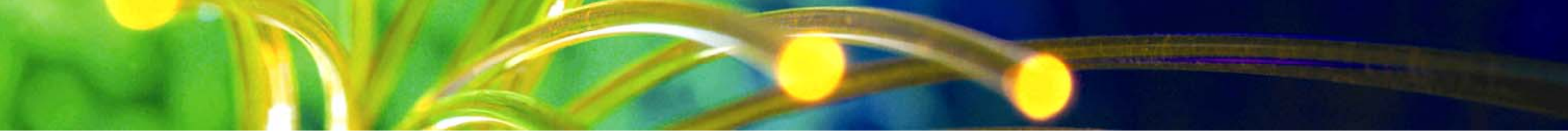
$$\mathbf{R}_n=\left\{\boldsymbol{\upsilon}\left|\ln\frac{f\left(\underline{\boldsymbol{\upsilon}}/\underline{\mathbf{s}}_n\right)}{f\left(\underline{\boldsymbol{\upsilon}}/\underline{\mathbf{s}}_m\right)}\geq 0\quad\forall m\neq n\right.\right\}$$

$$\mu_k=-\upsilon_k$$

$$d\mu_k=-d\upsilon_k$$

$$\ln\frac{f\left(\underline{\boldsymbol{\upsilon}}/\underline{\mathbf{s}}_n\right)}{f\left(\underline{\boldsymbol{\upsilon}}/\underline{\mathbf{s}}_m\right)}=\ln\prod_{x_{m,k=1}=0\neq x_{n,k}}\frac{f_0\left(-\upsilon_k\right)}{f_0\left(\upsilon_k\right)}+\ln\underline{\prod_{x_{m,k=1}=1\neq x_{n,k}}\frac{f_0\left(\upsilon_k\right)}{f_0\left(-\upsilon_k\right)}}=\ln\prod_{x_{m,k=1}\neq x_{n,k}}\underline{\frac{f_0\left(-\upsilon_k\right)}{f_0\left(\upsilon_k\right)}}$$

REMINDER: $D_m=D\quad\forall m$



We know that the pairwise probability (*AWGN*) is given by:

$$P_e(i \to j) = Q\left(\frac{\left\|\underline{s}_i - \underline{s}_j\right\|}{\sqrt{2No}}\right)$$

where:

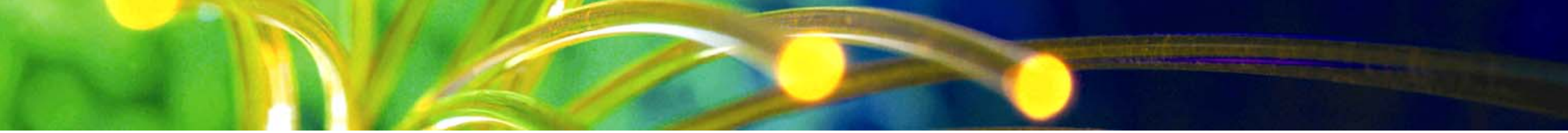
$$\left\|\underline{\mathbf{s}}_i - \underline{\mathbf{s}}_j\right\| = \left\|\underline{\mathbf{x}}_i - \underline{\mathbf{x}}_j\right\| = 2\sqrt{\omega\left(\underline{\mathbf{x}}_i \oplus \underline{\mathbf{x}}_j\right)\ E}$$

Hamming distance!

And, from the union-bound and considering $\underline{x}_1=\underline{0}$ as reference:

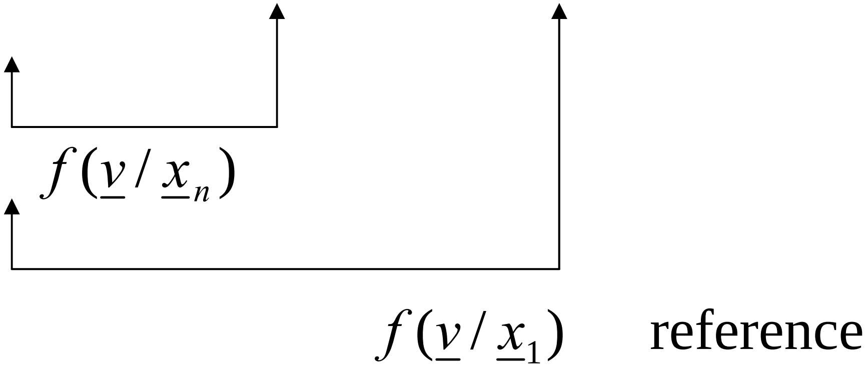
$$P(e) \le \sum_{m=2}^{M} Q\left(\frac{\left\|\underline{x}_m - \underline{x}_1\right\|}{\sqrt{2N_0}}\right) = \sum_{m=2}^{M} Q\left(\sqrt{2\frac{E_b}{N_o}\omega_k}\right)$$

extension to any binary-input symmetric channel



In general, from the Bhattacharyya-bound, we have that:

$$P_e(\underbrace{1}_{reference} \to n) \le \prod_{k=1}^{K} \int_{-\infty}^{+\infty} \sqrt{f(v/x_{n_k})\, f(v/x_{1_k} = +\sqrt{E})}\, dv =$$



$f(\underline{v}/\underline{x}_n)$

$f(\underline{v}/\underline{x}_1)$ reference

(only different elements)

$$= \prod_{\substack{k \\ x_{n_k} \neq x_{1_k}}} \int_{-\infty}^{+\infty} \sqrt{\underbrace{f(v/x_{n_k} = -\sqrt{E})}_{f_1(v)}\, \underbrace{f(v/x_{n_k} = +\sqrt{E})}_{f_0(v)}}\, dv \quad \Rightarrow \quad ../...$$

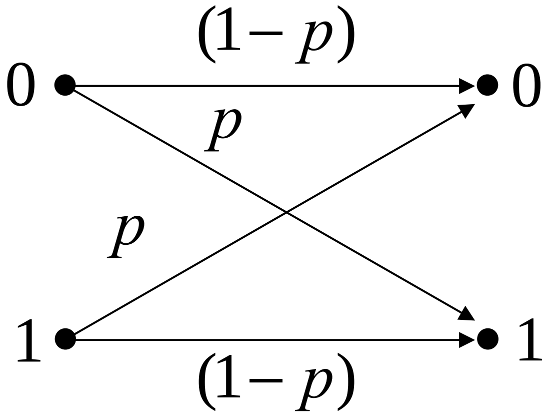
$$\Rightarrow P_e(1 \to n) \le \left(\int_{-\infty}^{+\infty} \sqrt{f_1(v) f_0(v)}\, dv \right)^{\omega_k}$$

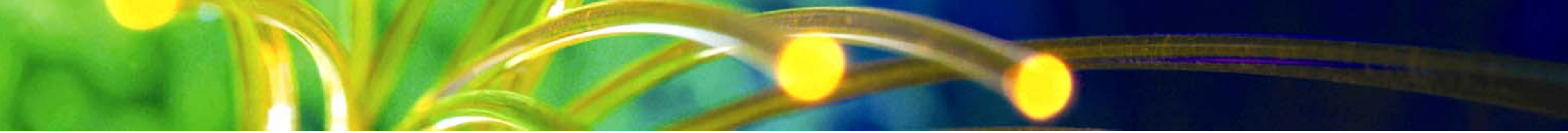
and:

$$P(e) \le \sum_{m=2}^{M} P_e(1 \to m) = \sum_{m=2}^{M} e^{\omega_k \ln \int_{-\infty}^{+\infty} \sqrt{f_1(v) f_0(v) dv}} = \sum_{m=2}^{M} e^{\omega_k \ln \sqrt{4p(1-p)}}$$

Binary symmetric channel (BSC)

(verify!!)





or, a simpler approximation:

$$P(e) < (M-1)Q\left(\sqrt{2\frac{E_b}{N_o}\min_{k\neq1}\omega_k}\right)$$

and:

$$P(e) < (M-1)e^{\min\limits_{k\neq1}\omega_k\left(\ln\int_{-\infty}^{+\infty}\sqrt{f_0(v)f_1(v)}dv\right)}$$