MINT-MERIT

Master of Science in Information and Communication Technologies

COMMUNICATION THEORY

FADING CHANNELS

Jaume Riba, Gregori Vázquez
Dept. of Signal Theory and Communications
Technical University of Catalonia
[jaume.riba;gregori.vazquez]@upc.edu

MEMORYLESS LINEAR TIME-INVARIANT GAUSSIAN CHANNELS

AWGN CHANNEL

$$y[n] = x[n] + w[n] \qquad w[n] \text{ distributed as } CN(0, N_o)$$

2-D Capacity:

$$C_{AWGN} = BW \log_2\left(1 + \frac{S}{N_o BW}\right) \text{ bits/sec.}$$

2-D Spectral Efficiency:

$$C_{AWGN} = \log_2\left(1 + \frac{S}{N_o BW}\right) \text{ bits/sec./Hz}$$

GENERAL COMMENTS

- $f(SNR) := \log(1+SNR)$ **is concave, that is,** $f''(x \le 0)$

- **Low and high SNR regimes:**

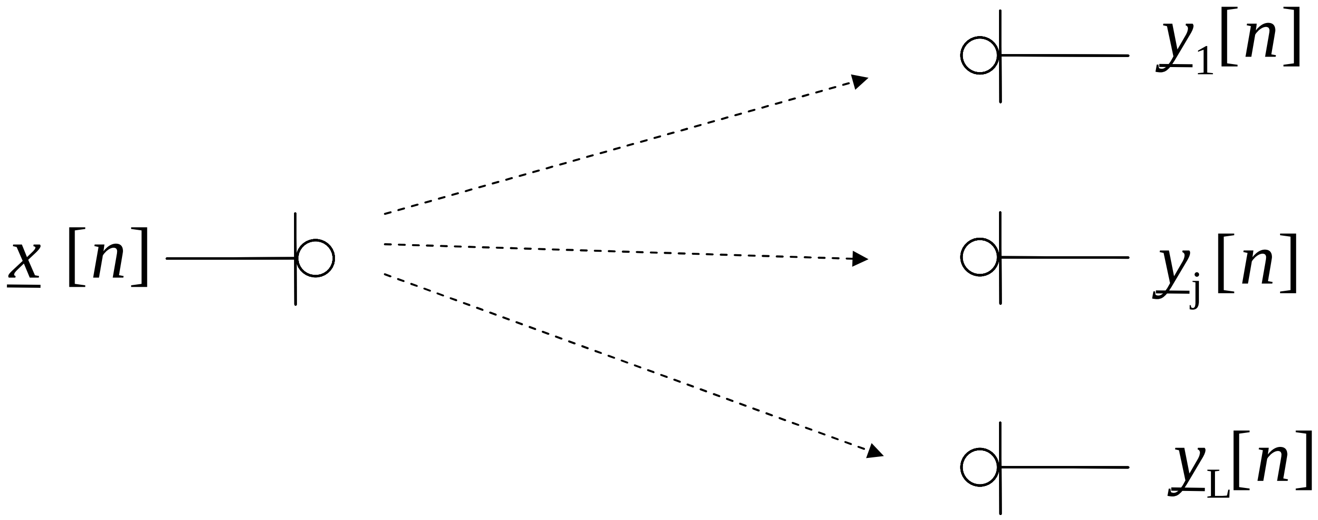
$$\log_2(1+SNR) \approx SNR \log_2 e \quad \text{for} \quad SNR \approx 0$$
$$\log_2(1+SNR) \approx \log_2 SNR \quad \text{for} \quad SNR >> 1$$

- **Wideband-regime (large *BW*→ *low SNR regime*):**

$$C_{AWGN} = BW \log_2\left(1+\frac{S}{N_o BW}\right) \text{bits/sec.} \approx BW\left(\frac{S}{N_o BW}\right)\log_2 e = \left(\frac{S}{N_o}\right)\log_2 e$$

$$\left.\frac{E_b}{N_o}\right|_{\min} = \frac{1}{\log_2 e} \approx -1.59 dB$$

LTI GAUSSIAN CHANNELS (SIMO) → RX beamforming



$$y_l[n] = h_l x[n] + w_l[n] \quad ; l = 1,2,\ldots,L \quad ; w[n] \text{ i.i.d. as } CN(0, N_o)$$

or:

$$\underline{\mathbf{y}}(n) = \underline{\mathbf{h}}x[n] + \underline{\mathbf{w}}(n)$$

for :

$$\underline{\mathbf{h}} = [h_1 \ h_2 \ldots h_L]^T, \quad \underline{\mathbf{y}} = [y_1 \ y_2 \ldots y_L]^T, \quad \underline{\mathbf{w}} = [w_1 \ w_2 \ldots w_L]^T$$

- The optimal MAP detection provides a sufficient statistic:

$$\upsilon(n) = \frac{\underline{\mathbf{h}}^H}{\|\underline{\mathbf{h}}\|}\underline{\mathbf{y}}(n) = \|\underline{\mathbf{h}}\| x(n) + \frac{\underline{\mathbf{h}}^H}{\|\underline{\mathbf{h}}\|}\underline{\mathbf{w}}(n)$$

for :

$$\underline{\mathbf{h}} = [h_1\ h_2 \ldots h_L]^T, \quad \underline{\mathbf{y}} = [y_1\ y_2 \ldots y_L]^T, \quad \underline{\mathbf{w}} = [w_1\ w_2 \ldots w_L]^T$$

- RX with a perfect *CSI*.

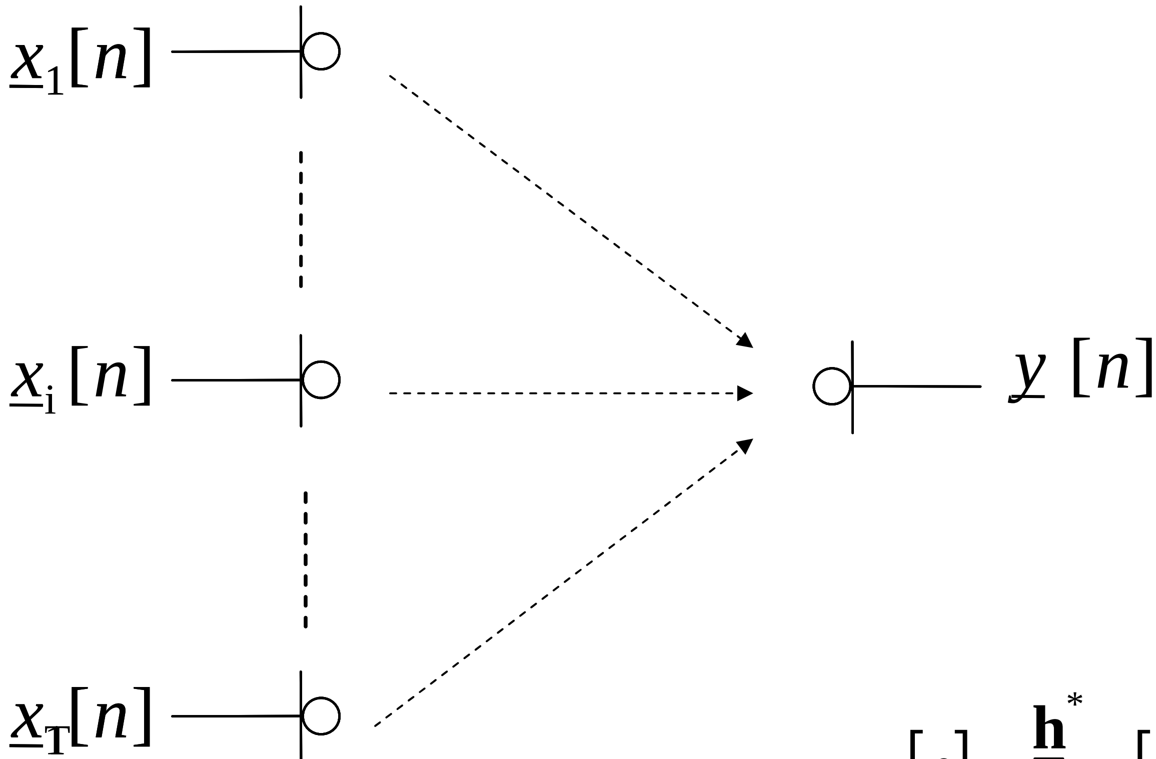
- The normalized detector provides a noise mean power:

$$E\left[|\tilde{w}(n)|^2\right] = E\left[\left|\frac{\underline{\mathbf{h}}^H}{\|\underline{\mathbf{h}}\|}\underline{\mathbf{w}}(n)\right|^2\right] = N_0$$

- Capacity per channel-use becomes (*receive beamforming*):

$$C = \log_2\left(1 + \|\underline{\mathbf{h}}\|^2 \frac{S}{N_o}\right) \quad \text{bits/s/Hz}$$

LTI GAUSSIAN CHANNELS (MISO) → TX beamforming



$$\underline{\mathbf{x}}_T[n] = \frac{\underline{\mathbf{h}}^*}{\|\underline{\mathbf{h}}\|} x[n]$$

- We notive that:

$$S_T = E\left[\|\underline{\mathbf{x}}_T[n]\|^2\right] = E\left[\|x[n]\|^2\right] = S$$

• At the receiver (channel contibution):

$$y(n) = \underline{\mathbf{h}}^T \underline{\mathbf{x}}_T[n] + w[n] = \|\underline{\mathbf{h}}\| x[n] + w[n]$$

• TX with a perfect *CSI* → *<u>need of a feedback-channel</u>*.

• More complex TX and impact of calibration.

• Capacity per channel-use becomes (*transmit beamforming*):

$$C = \log_2\left(1 + \|\underline{\mathbf{h}}\|^2 \frac{S}{N_o}\right) \quad \text{bits/s/Hz}$$

• Transmit and receive beamforming provide the same equivalent scalar capacity for a perfect *CSI* at TX or RX.

FREQUENCY-SELECTIVE LINEAR TIME-INVARIANT GAUSSIAN CHANNELS

FREQUENCY-SELECTIVE CHANNELS (SVD/OFDM)

$$y[n] = \sum_{l=0}^{L-1} h_j x[n-l] + w[n] \qquad w[n] \text{distributed as } CN(0, N_o)$$

• Channel matrix *SVD* or *OFDM* transmission are two alternatives for providing an equivalent memoryless parallel channel scheme, such that:

$$\tilde{y}_k[n] = \tilde{h}_k x_k[n] + \tilde{w}_k[n] \quad k = 0,1,\ldots, N_c - 1$$

• *SVD* requires *CSI* both at the TX and the RX sides whereas for OFDM only at the receiving end.

• Guard-times are required to avoid *IBI*. In particular, in OFDM the cyclic-prefix implies a penalty in the transmitted power efficiency.

• Under transmitted power constraints, the waterfilling power allocation maximizes the total capacity:

$$C = \sum_{k=0}^{N_c-1} \log_2\left(1 + \left|\tilde{h}_k\right|^2 \frac{S_k}{N_o}\right) \text{bits/sec./Hz}$$

REMARK:

• Coding *across channels* *will not help if channel coding is performed in large blocks but it can still help for reducing the decoding error probability for finite lengths as it allows to virtually increase the length of the blocks.*

SLOW FADING (GAUSSIAN) CHANNELS

SLOW FADING CHANNEL: Outage-Probability

$$y[n] = h[n]x[n] + w[n] \qquad w[n]\text{distributed as } CN(0, N_o)$$

$$E\left[\left|h[n]\right|^2\right] = 1$$

• The random nature of the channel implies that it will be in *outage* for transmission rates *R* bits/s/Hz:

$$\log_2\left(1 + |h|^2 SNR\right) < R$$

• *Concept makes sense if the channel is changing very slowly*.

• The channel *outage probablity* is defined as:

$$\boxed{P_{out} := PROB\left\{\log_2\left(1 + |h|^2 SNR\right) < R\right\}}$$

- For a Rayleigh fading channel:

$$P_{out} := 1 - \exp\left(-\frac{2^R - 1}{SNR} \right)$$

- In a high-SNR regime:

$$P_{out} \approx \frac{2^R - 1}{SNR} \propto \frac{1}{SNR}$$

- It is observed a slow decaying of the outage probability with the SNR.

- *It is possible to conclude that coding can not combat the impact of the channel fades either at high and low-SNRs.*

SLOW FADING CHANNEL: ε-Outage Capacity

• The definition of ε-outage capacity C_ε comes from considering the largest transmission rate *R* such that the outage-probability is below *ε*:

$$C_\varepsilon := \log_2\left(1 + F^{-1}(1-\varepsilon)SNR\right) \text{bits/s/Hz}$$

$$\text{where:} \quad F(x) := PROB\left\{|h|^2 > x\right\}$$

• High-SNR:

$$C_\varepsilon^{high} \approx \log_2(SNR) + \log_2\left(F^{-1}(1-\varepsilon)\right) \approx C_{AWGN}^{high} - \log_2\left(\frac{1}{F^{-1}(1-\varepsilon)}\right)$$

It does not depend on the SNR

• Low-SNR:

$$C_\varepsilon^{low} \approx F^{-1}(1-\varepsilon)SNR\log_2 e \approx F^{-1}(1-\varepsilon)C_{AWGN}^{low}$$

• For small maximum outage probability ε:

$$F^{-1}(1-\varepsilon) \approx \varepsilon$$

$$C_{\varepsilon}^{high} \approx C_{AWGN}^{high} - \log_2\left(\frac{1}{\varepsilon}\right)$$

$$C_{\varepsilon}^{low} \approx \varepsilon\, C_{AWGN}^{low}$$

$$\boxed{\frac{C_{\varepsilon}^{low}}{C_{AWGN}^{low}} \approx \varepsilon}$$

SOLUTIONS?

SLOW FADING CHANNEL: TX or RX-Diversity

- From:

$$P_{out}^{RX}\left(R\right)=P_{out}^{TX\ \ full-CSI}\left(R\right)=PROB\left\{\log_2\left(1+\left\|\underline{\mathbf{h}}\right\|^2 SNR\right)< R\ \right\}$$

or:

$$P_{out}^{RX}\left(R\right)=P_{out}^{TX\ \ full-CSI}\left(R\right)=PROB\left\{\ \left\|\underline{\mathbf{h}}\right\|^2 < \frac{2^R-1}{SNR}\ \right\}$$

Reminder: *For a Rayleigh channel, the square of the norm of the channel vector includes the sum of squares for 2L independent Gaussian terms and it is then distributed as Chi-Square with 2L degrees of freedom.*

$$f\left(x\right)=\frac{1}{\left(L-1\right)!}x^{L-1}e^{-x} \qquad x\geq 0$$

• High-*SNR*:

$$P_{out}^{RX}(R) = P_{out}^{TX\ full-CSI}(R) = PROB\left\{ \|\underline{\mathbf{h}}\|^2 < \frac{2^R - 1}{SNR} \right\} \approx \frac{\left(2^R - 1\right)^L}{L!\, SNR^L}$$

$$\boxed{P_{out}^{RX}(R) = P_{out}^{TX\ full-CSI}(R) \propto \frac{1}{SNR^L}}$$

• Low-*SNR*:

$$C_\varepsilon^{RX} = C_\varepsilon^{TX\ full-CSI} = F^{-1}(1-\varepsilon)C_{AWGN}^{low} \approx (L!)^{1/L}(\varepsilon)^{1/L} SNR \log_2 e$$

$$F(x) := PROB\left\{ \|\underline{\mathbf{h}}\|^2 > x \right\}$$

$$\boxed{C_\varepsilon^{RX} = C_\varepsilon^{TX\ full-CSI} \propto (\varepsilon)^{1/L}}$$

Note: $F(1-\delta) = PROB\left\{ \|\underline{\mathbf{h}}\|^2 < \delta \right\} \approx \frac{1}{L!}\delta^L$; small $\delta \Rightarrow F^{-1}(1-\delta) \approx (L!\delta)^{1/L}$

SLOW FADING CHANNEL: No-CSI at the TX → Alamouti

• For simplicity let's consider L=2. *NO CSI AVAILABLE AT THE TX*. Alamouti implies a 3dB loss with respect to transmit-beamforming:

$$C_{Alam} = \log_2\left(1 + \left\|\underline{\mathbf{h}}\right\|^2 \frac{S}{2N_o}\right) \quad \text{bits/s/Hz}$$

$$P_{out}^{Alam}(R) = PROB\left\{ \log_2\left(1 + \left\|\underline{\mathbf{h}}\right\|^2 \frac{SNR}{2}\right) < R \right\}$$

REMARK: The result can be generalized to larger number of elements L as Alamouti implies an isotropic radiation pattern. Number of Alamouti schemes is limited in L.

Outage in no-TX CSI Parallel Independent Channels

<u>*UNIFORM POWER ALLOCATION*</u>

T_c time − slots

L **carriers**

$$y_k[n] = h_k x_k[n] + w_k[n] \quad k = 1,2,\ldots,L;\ n = 1,2,\ldots,T_c$$

T_c defines the minimum *coherence-time* of the *L* channels

• Reliable communications are possible for:

$$R_{coherent-slot}^{per-subchannel} < \frac{1}{L}\sum_{k=1}^{L}\log_2\left(1+|h_k|^2 SNR\right) \textbf{bits/sec/Hz}$$

or it will be in outage, otherwise for:

$$P_{out}^{parall}(R) = PROB\left\{ \frac{1}{L}\sum_{k=1}^{L}\log_2\left(1+|h_k|^2 SNR\right) < R \right\}$$

FAST FADING (GAUSSIAN) CHANNELS

FAST FADING CHANNEL

- *The Fast Fading Regime is defined as for those working conditions in which codewords spans along many channel coherence-time periods.*

- Block-fading model is such channel does not change during T_c seconds:

$$y[n] = h_l x[n] + w[n] \quad ; l-th \text{ coherence-time period.}$$

- For a large coherence-time T_c>>1, we model the problem as for transmitting in L independent, fading channels and outage probability becomes:

$$P_{out}^{block-fad}(R) = PROB\left\{ \frac{1}{L}\sum_{l=1}^{L}\log_2\left(1+|h_l|^2 SNR\right) < R \right\}$$

- But, outage becomes a random variable. For large L (and probably after randomizing with an interleaver to avoid prohibitive code-lengths) and for *fast-fading (i.i.d. channel gains)* we can make use of the law of large numbers, such that:

$$C_{block-fad} = C_{fast-fad} = E\left[\log_2\left(1+|h|^2 SNR\right)\right]$$

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• Low-*SNR*:

$$C_{block-fad} = C_{fast-fad} \approx E\left[|h|^2 SNR\right] \log_2 e = SNR \log_2 e = C_{AWGN}$$

$$C_{block-fad} = C_{fast-fad} = C_{AWGN}$$

• High-*SNR*:

$$C_{block-fad} = C_{fast-fad} \approx E\left[\log_2\left(|h|^2 SNR\right)\right] =$$

$$= \log_2(SNR) + E\left[\log_2\left(|h|^2\right)\right] = C_{AWGN} + E\left[\log_2\left(|h|^2\right)\right]$$

$$C_{block-fad} = C_{fast-fad} \approx C_{AWGN} - 0.83 \text{ bits/s/Hz}$$

2.5dB power-loss

In general: $$C_{block-fad} = C_{fast-fad} \leq \log_2\left(1 + E\left[|h|^2 SNR\right]\right) = C_{AWGN}$$

WATERFILLING IN FAST-FADING CHANNELS

$$\max_{\{P_l\}} \frac{1}{L}\sum_{l=1}^{L}\log_2\left(1+\frac{P_l\left|h_l\right|^2}{N_o}\right)$$

$$\text{subject to:} \quad \frac{1}{L}\sum_{l=1}^{L}P_l = P$$

$$\longrightarrow \quad \frac{1}{L}\sum_{l=1}^{L}\left(\frac{1}{\lambda}-\frac{N_o}{\left|h_l\right|^2}\right)^+ = P$$

$$\downarrow \; L \gg$$

$$E\left[\left(\frac{1}{\lambda}-\frac{N_o}{\left|h\right|^2}\right)^+\right] = P$$

$$P(h) = \left(\frac{1}{\lambda}-\frac{N_o}{\left|h\right|^2}\right)^+$$

$$C = E\left[\log_2\left(1+\frac{P(h)\left|h\right|^2}{N_o}\right)\right]$$

CHANNEL PREDICTION/TRACKING