APPENDIX A: THE AUTOCORRELATION MATRIX

There are many ways to introduce this theory. We will consider the power output to a *WSS* (*Wide-Sense Stationary*) input for a *FIR* filter:

$$\underline{w} \equiv [w_0 \;\; w_1 \;\; w_2 \;\; \ldots \;\; w_{\mathrm{N-1}}]^T \qquad \underline{x}(u) \equiv [x(u) \;\; \mathrm{x(u-1)} \;\; \mathrm{x(u-2)} \;\; \ldots \;\; x(\mathrm{u-N+1})]^T$$

$$\boxed{y(u) = \underline{w}^H \, \underline{x}(u)}$$

('*H*' denotes transpose-conjugate for complex data and just only transpose for real data)

The filter output power is then given by:

$$p_y \equiv E\left[|y(u)|^2 \right] = E\left[y(u) y^*(u) \right] = \underline{w}^H E\left[\underline{x}(u) \underline{x}^H(u) \right] \underline{w} \equiv \underline{w}^H \underline{\underline{R}} \, \underline{w}$$

where:

$$\underline{\underline{R}} \equiv E\left[\underline{x}(u) \underline{x}^H(u) \right] \qquad \text{Autocorrelation Matrix}$$

Thus:

$$\underline{\underline{R}} = E\left[\underline{x}(u)\underline{x}^H(u)\right] = \begin{bmatrix} R_{xx}(0) & R_{xx}(1) & \cdots & R_{xx}(N-1) \\ R_{xx}^*(1) & R_{xx}(0) & \cdots & \vdots \\ R_{xx}^*(2) & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & R_{xx}(1) \\ R_{xx}^*(N-1) & \cdots & \cdots & R_{xx}(0) \end{bmatrix}$$

<u>Toeplitz</u> matrix

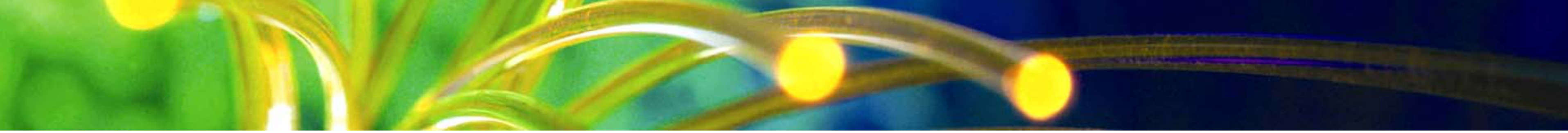
We see also that:

$$P_y \geq 0 \quad \forall \underline{w} \quad \Rightarrow \quad \underline{w}^H \underline{\underline{R}}\,\underline{w} \geq 0 \quad \text{(real)} \quad \forall \underline{w}$$

<u>Semi-positive</u> matrix

and:

$$\underline{\underline{R}}^H = \underline{\underline{R}} \quad \underline{\text{Hermitic}} \text{ matrix} \longrightarrow$$ *The eigenstructure is very useful*



We now consider:

$$\underline{\underline{R}}\,\underline{q} = \lambda \underline{q} \quad \text{with} \quad \|\mathrm{q}\|^2 = \underline{q}^H \underline{q} = 1$$

1.- $\lambda \geq 0$ and real

$$\text{From } \underline{\underline{R}}\,\underline{q} = \lambda \underline{q} \Rightarrow \underline{q}^H \underline{\underline{R}}\,\underline{q} = \lambda \underline{q}^H \underline{q} = \lambda \geq 0 \ !!$$

Real!!

$$2. - \left.\begin{array}{l} \underline{\underline{\mathrm{R}}}\,\underline{\mathrm{q}}_{\mathrm{i}} = \lambda_{\mathrm{i}} \underline{\mathrm{q}}_{\mathrm{i}} \\ \underline{\underline{\mathrm{R}}}\,\underline{\mathrm{q}}_{\mathrm{j}} = \lambda_{\mathrm{j}} \underline{\mathrm{q}}_{\mathrm{j}} \end{array}\right\} \text{ with } \lambda_i \neq \lambda_j \Rightarrow \underline{\mathrm{q}}_{\mathrm{i}}^{\mathrm{H}} \underline{\mathrm{q}}_{\mathrm{j}} = \delta_{\mathrm{ij}} \ \text{(orthonormal)}$$

$$\left.\begin{array}{l} \underline{q}_j^H \underline{\underline{R}}\,\underline{q}_i = \lambda_i \underline{q}_j^H \underline{q}_i \\ \underline{q}_i^H \underline{\underline{R}}\,\underline{q}_j = \lambda_j \underline{q}_i^H \underline{q}_j \end{array}\right\} \quad 0 = (\lambda_i - \lambda_j) \underline{q}_j^H \underline{q}_i$$

$$\lambda_i \neq \lambda_j \Rightarrow \underline{q}_j^H \underline{q}_i = 0$$

↑

λ_i, λ_j real

3.- $\underline{\underline{Q}} \equiv \left[\underline{q}_1 \;\; \underline{q}_2 \;\; \dots \;\; \underline{q}_N \right]$

$\underline{\underline{\Lambda}} \equiv diag\left[\lambda_1 \;\; \lambda_2 \;\; \dots \;\; \lambda_N \right]$

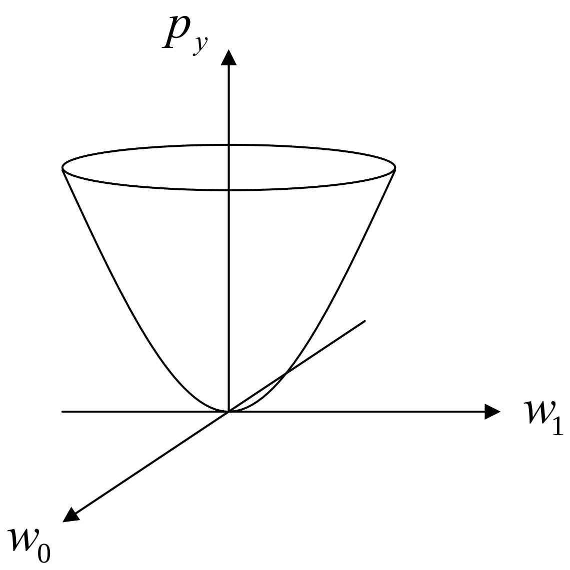
$$\boxed{\underline{\underline{\mathrm{R}}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^H}$$

with $\underline{\underline{Q}}\,\underline{\underline{Q}}^H = \underline{\underline{Q}}^H \underline{\underline{Q}} = \underline{\underline{I}}$ Unitary

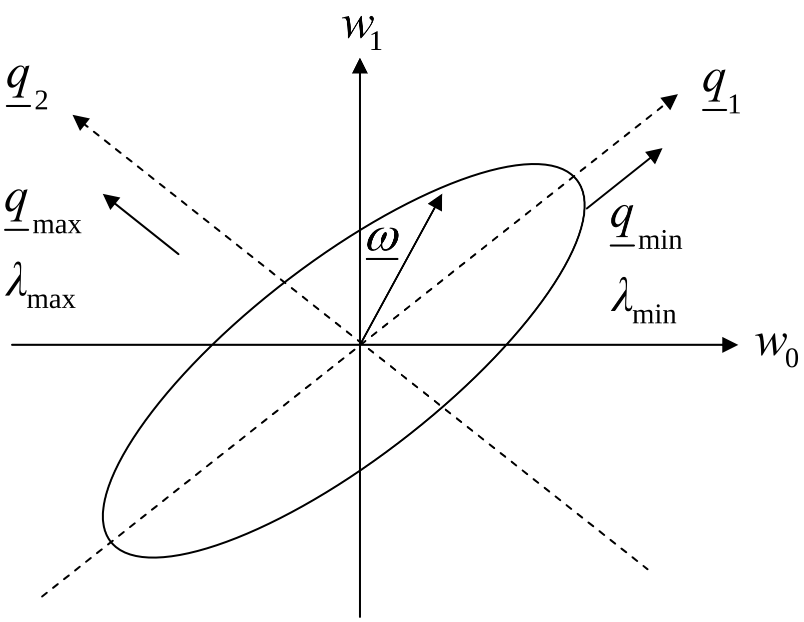
Or:

$$\boxed{\underline{\underline{R}} = \sum_{i=1}^{N} \lambda_i \underline{q}_i \underline{q}_i^{\,H}}$$

4.- $\mathrm{P_y} = \underline{w}^{\mathrm{H}} \underline{\underline{\mathrm{R}}}\,\underline{w} = \underline{w}^H \sum_{i=1}^{N} \lambda_i \underline{q}_i \underline{q}_i^{\,H} \underline{w} = \sum_{i=1}^{N} \lambda_i \underline{w}^H \underline{q}_i \underline{q}_i^{\,H} \underline{w} = \sum_{i=1}^{N} \lambda_i \left\| \underline{w}^H \underline{q}_i \right\|^2$



$$P_y = \sum_{i=1}^{N} \lambda_i \left\| \underline{w}^H \underline{q}_i \right\|^2$$



Or, in other words:

$$\left.\begin{array}{c}\max \\ \min\end{array}\right\} \quad \underline{w}^H \underline{\underline{R}}\, \underline{w}$$

Subject to:

$$\underline{w}^H \underline{w} = 1$$

APPENDIX B: THE SINGULAR VALUE DECOMPOSITION OF A MATRIX (SVD)

We saw in *appendix A* that for a hermitic NxN matrix there exists an orthonormal (unitary) matrix $\underline{\underline{Q}}$ such that:

$$\underline{\underline{R}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^H \quad \text{(Schur decomposition)}$$

With

$$\underline{\underline{Q}}^H \underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^H = \underline{\underline{I}}$$

$$\underline{\underline{\Lambda}} = diag(\lambda_1 \;\; \lambda_2 \;\; \ldots \;\; \lambda_N) \qquad \lambda_i \geq 0$$

Now, we will consider an MxN matrix $\underline{\underline{H}}$ of rank r($\underline{\underline{H}}$)=$\underline{\underline{R}}$>0

Then, there exist an MxR matrix $\underline{\underline{S}}$ and an NxR matrix $\underline{\underline{T}}$ such that:

$$\left.\begin{array}{l} \underline{\underline{S}}^H \underline{\underline{S}} = \underline{\underline{I}} \\ \underline{\underline{T}}^H \underline{\underline{T}} = \underline{\underline{I}} \end{array}\right\} \quad \text{with} \qquad \underline{\underline{H}} = \underline{\underline{S}}\,\underline{\underline{\Lambda}}\,\underline{\underline{T}}^H \qquad \text{(SVD)}$$

$$\underline{\underline{\Lambda}} = diag(\lambda_1 \;\; \lambda_2 \;\; \ldots \;\; \lambda_N) \qquad \lambda_i \geq 0$$

(see ref. [3] for details)