MSICT

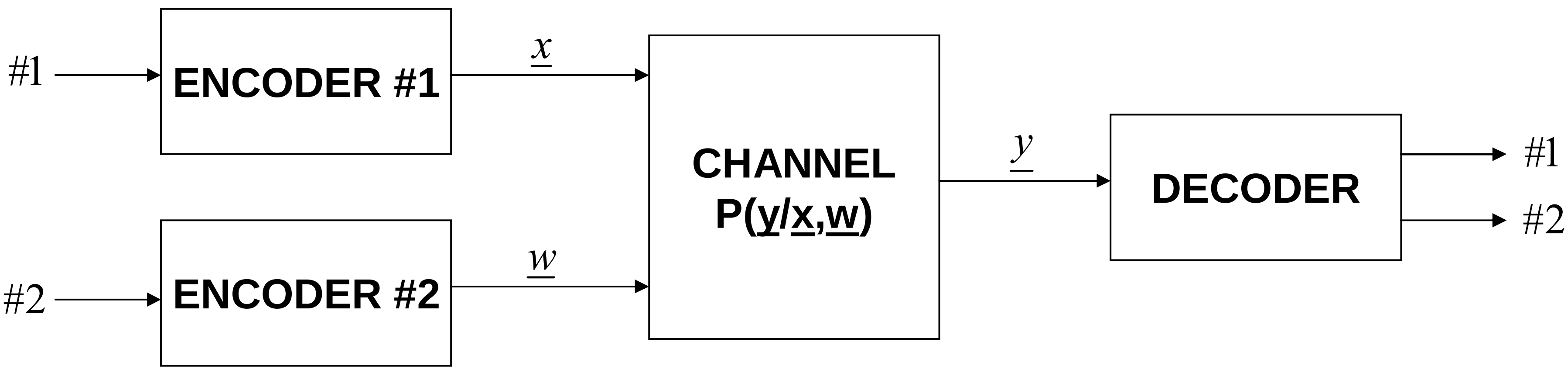
Master of Science in Information and Communication Technologies

COMMUNICATION THEORY

MULTIPLE ACCESS

Gregori Vázquez Grau
Dept. of Signal Theory and Communications
Technical University of Catalonia
gregori@gps.tsc.upc.edu

MULTIPLE-ACCESS CHANNELS



Alphabets: A_x, A_w, A_y

$$\left.\begin{array}{l}\underline{x} = (x_1, x_2, \ldots, x_N) \\ \underline{w} = (w_1, w_2, \ldots, w_N)\end{array}\right| \longrightarrow \text{Block coding} \longrightarrow \begin{cases}\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_M\} & M \\ \{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_L\} & L\end{cases}$$

For a *DMC*, we have that:

$$P(\underline{y}/\underline{x},\underline{w}) = \prod_{n=1}^{N} P(y_n / x_n, w_n)$$

and the rates for the two sources are given by:

$$R_1 = \frac{\ln M}{N} \qquad\qquad R_2 = \frac{\ln L}{N}$$

From Ahlswede-Liao:

$$\boxed{\begin{aligned} &R_1 + R_2 \le I(x, w; y) \\ &0 \le R_1 \le I(x; y / w) \\ &0 \le R_2 \le I(w; y / x) \end{aligned}}$$

From the definition of mutual (*average*) information:

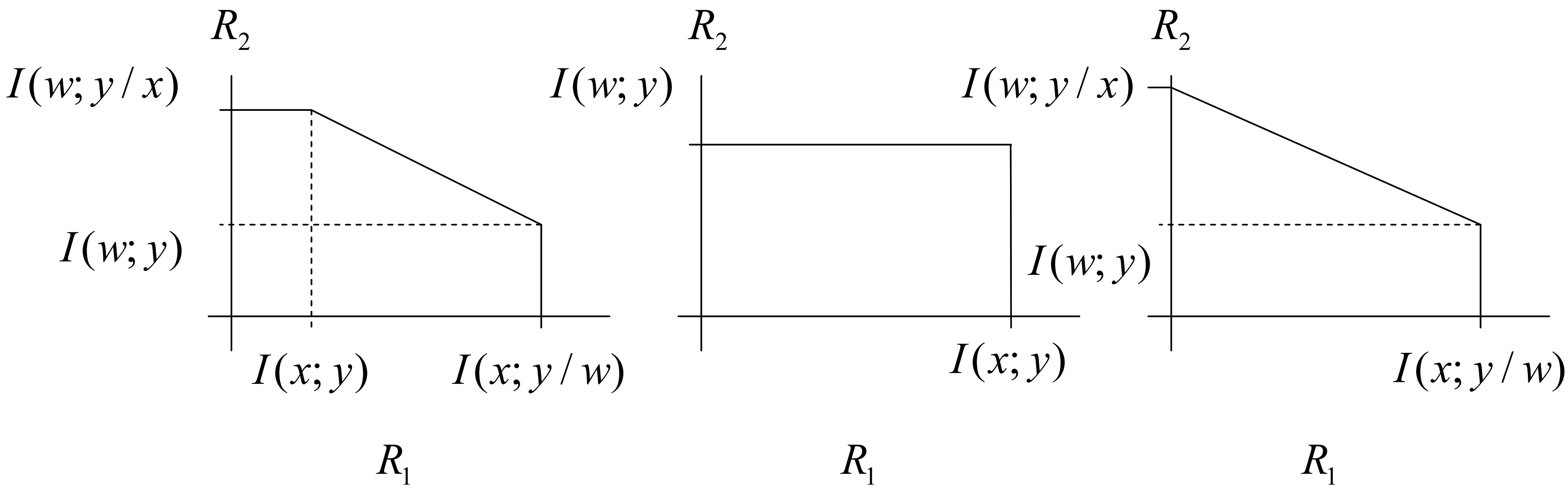
$$I(A^x; A^y) \equiv E\left[I(x;y)\right] = \sum_x \sum_y P(x;y) \ln \frac{P(y/x)}{P(y)}$$

we have that:

$$\boxed{\begin{aligned} I(x,w;y) &\equiv \sum_x \sum_w \sum_y P(x,w;y) \ln \frac{P(y/x,w)}{P(y)} = \\ &= \sum_x \sum_w \sum_y P(y/x,w) P_1(x) P_2(w) \ln \frac{P(y/x,w)}{P(y)} \end{aligned}}$$

$$\boxed{I(x;y/w) \equiv \sum_x \sum_w \sum_y P(y/x,w) P_1(x) P_2(w) \ln \frac{P(y/x,w)}{P(y/w)}}$$

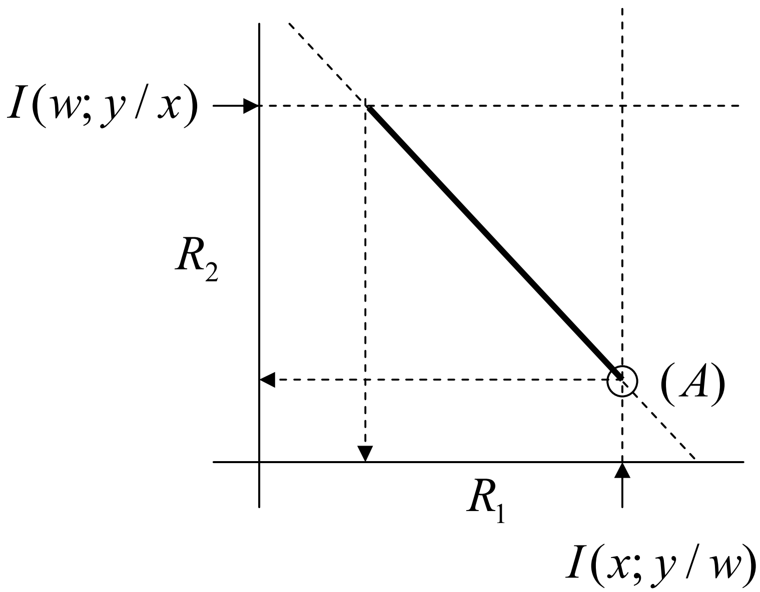
$$\boxed{I(w;y/x) \equiv \sum_x \sum_w \sum_y P(y/x,w) P_1(x) P_2(w) \ln \frac{P(y/x,w)}{P(y/x)}}$$



$$I(x;y) = I(x;y/w) \qquad\qquad I(x;y) = 0$$

Notice that for x & w independent:

$$I(x;y/w) \geq I(x;y) \qquad ; \qquad I(w;y/x) \geq I(w;y)$$



In (A): $R_1 = I(x;y/w)$

We know that in (A)

$$R_1 + R_2 = I(x,w;y)$$

and thus:

$$I(x;y/w) + R_2 = I(x,w/y)$$

or:

$$R_2 = I(x,w;y) - I(x;y/w)$$

$$R_2 = \sum_{xy\omega} P(x,w,y)\ln\frac{P(y/x,w)}{P(y)} - \sum_{x\omega y} P(x,w,y)\ln\frac{P(y/x,w)}{P(y/w)} =$$

$$= \sum_{x\omega y} P(x,w,y)\ln\frac{P(y/w)}{P(y)} = I(w;y)$$

For the AWGN channel:

$$y = x + w + \underbrace{z}_{Noise} \qquad z \sim N(0, \sigma^2)$$

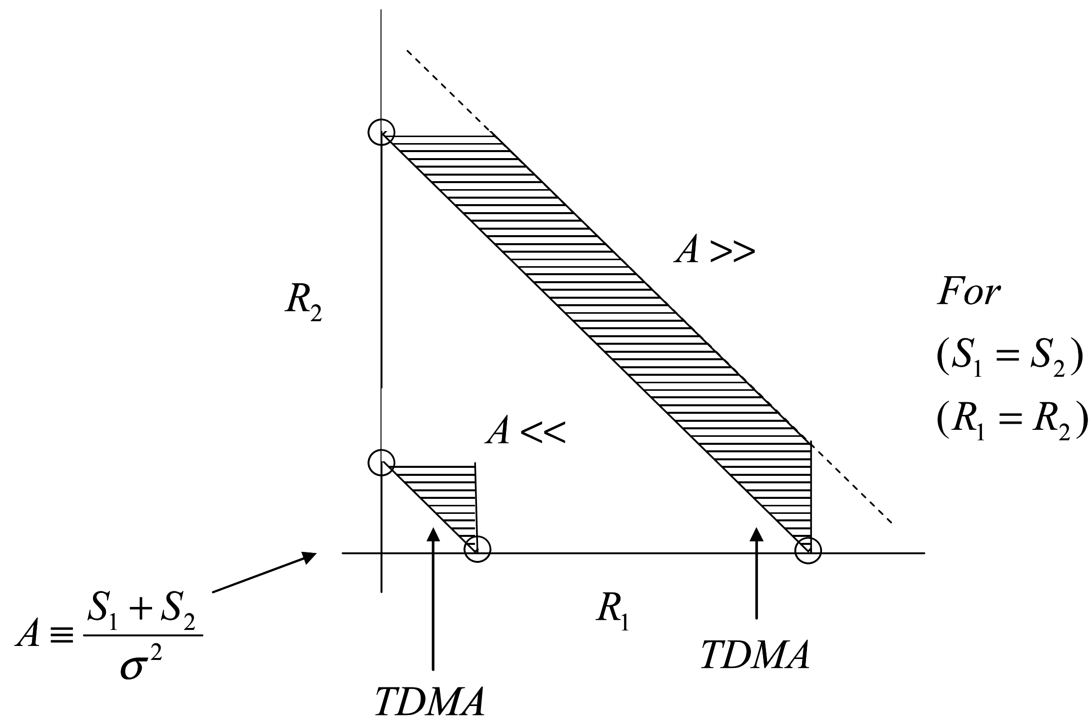
$$I(x, w; y) \le \frac{1}{2}\log\left(1 + \frac{S_1 + S_2}{\sigma^2}\right)$$

$$I(x; y/w) \le \frac{1}{2}\log\left(1 + \frac{S_1}{\sigma^2}\right)$$

$$I(w; y/x) \le \frac{1}{2}\log\left(1 + \frac{S_2}{\sigma^2}\right)$$

$$S_1 = E[x^2]$$

$$S_2 = E[\omega^2]$$



Consider the following multiple access scheme:

$$\underline{y} = \sum_{i=1}^{K} \sqrt{p_i}\, x_i \underline{\phi}_i + \underline{w}$$

The *linear detector* for user#1 will be given by:

$$\hat{x}_1 = \underline{\phi}_1^H \underline{y} = \underbrace{\sqrt{p_1}\, x_1 \underline{\phi}_1^H \underline{\phi}_1}_{User\,\#1} + \underbrace{\sum_{i=2}^{K} \sqrt{p_i}\, x_i \underline{\phi}_1^H \underline{\phi}_i}_{\text{Multiple-access Interference (MAI)}} + \underbrace{\underline{\phi}_1^H \underline{w}}_{\text{Noise}}$$

Depending on the signature set, $\{\phi_i\}_{1 \le i \le K}$ the '*Multiple Access Interference*' (*MAI*) can be a significant impairment for achieving a reliable communication.

We can try to optimize the linear detector.

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We now consider the more *general linear scheme*:

$$\hat{x}_1 = \underline{\omega}_1^H \underline{y} = \sqrt{p_1} x_1 \underline{\omega}_1^H \underline{\phi}_1 + \sum_{i=2}^{K} \sqrt{p_i} x_i \underline{\omega}_1^H \underline{\phi}_i + \underline{\omega}_1^H \underline{w}$$

such that:

$$\boxed{\max_{\underline{\omega}_1} \; SINR \equiv \frac{p_1 \; E\left[\left|x_1\right|^2\right] \quad \left|\underline{\omega}_1^H \underline{\phi}_1\right|^2}{\sum_{i=2}^{K} p_i \; E\left[\left|x_i\right|^2\right] \; \left|\underline{\omega}_1^H \underline{\phi}_i\right|^2 + E\left|\underline{\omega}_1^H \underline{w}\right|^2}}$$

or:

$$\boxed{\min_{\underline{\omega}_1} \; E\left[\left|x_1 - \hat{x}_1\right|^2\right] = E\left[\left|x_1 - \underline{\omega}_1^H \underline{y}\right|^2\right]}$$

We can also write the signal model as follows:

$$\boxed{\underline{y} = \sqrt{p_1}\, x_1 \underline{\phi}_1 + \underline{z}}$$

Where:

$$\underline{z} \equiv \sum_{i=2}^{K} \sqrt{p_i}\, x_i \underline{\phi}_i + \underline{w} \qquad (\equiv MAI + Noise)$$

In general, z is NOT considered white because of the MAI, but we can simplify the problem considering that z is white. Under this assumption:

$$\hat{x}_1 = \frac{1}{\underline{\phi}_1^H \underline{\phi}_1} \underline{\phi}_1^H \underline{y} \qquad \text{(conventional matched-filter)}$$

that is, the MAP/ML optimal detector corresponds to the projection of the observation on the desired user signature.

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We see that $\underline{z}$ is not white:

$$\underline{z} \equiv \sum_{i=2}^{K} \sqrt{p_i}\, x_i \underline{\phi}_i + \underline{w} \equiv \underline{\underline{\Phi}}_1 \underline{\underline{P}}_1^{1/2} \underline{x}_1 + \underline{w}$$

where:

$$\underline{\underline{\Phi}}_1 \equiv \begin{bmatrix} \underline{\phi}_2 & \underline{\phi}_3 & \cdots & \underline{\phi}_K \end{bmatrix}$$

$$\underline{\underline{P}}_1^{1/2} \equiv diag\begin{bmatrix} \sqrt{p_2} & \sqrt{p_3} & \cdots & \sqrt{p_K} \end{bmatrix}$$

$$\underline{x}_1 \equiv [x_2, x_3, \ldots, x_K]^T$$

and:

$$\underline{\underline{\Sigma}} \equiv E[\underline{z}\,\underline{z}^H] = \underline{\underline{\Phi}}_1 \underline{\underline{P}}_1^{1/2} \overbrace{E[\underline{x}_1 \underline{x}_1^H]}^{\underline{\underline{I}}} \underline{\underline{P}}_1^{1/2} \underline{\underline{\Phi}}_1^H + E[\underline{w}\,\underline{w}^H] = \underline{\underline{\Phi}}_1 \underline{\underline{P}}_1 \underline{\underline{\Phi}}_1^H + \sigma^2 \underline{\underline{I}}$$

The eigendecomposition of Σ is given as usual:

$$\boxed{\underline{\underline{\Sigma}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^H}$$

Thus, if we now take the following projection of the *MAI+Noise* term:

$$\underline{\tilde{z}} \equiv \underline{\underline{\Lambda}}^{-1/2}\underline{\underline{Q}}^H\underline{z} \quad \Rightarrow \quad E[\underline{\tilde{z}}\,\underline{\tilde{z}}^H] = \underline{\underline{\Lambda}}^{-1/2}\underline{\underline{Q}}^H E[\underline{z}\,\underline{z}^H]\underline{\underline{Q}}\,\underline{\underline{\Lambda}}^{-1/2} = \underline{\underline{\Lambda}}^{-1/2}\underline{\underline{Q}}^H\underline{\underline{\Sigma}}\,\underline{\underline{Q}}\,\underline{\underline{\Lambda}}^{-1/2} = \underline{\underline{I}} \quad !!$$

$$\underline{\tilde{z}} \ \text{ becomes white !!}$$

In other words, we can think on the equivalent signal model:

$$\boxed{\underline{\tilde{y}} = \underline{\underline{\Lambda}}^{-1/2}\underline{\underline{Q}}^H\underline{y} = \sqrt{p_1}\,x_1\underline{\underline{\Lambda}}^{-1/2}\underline{\underline{Q}}^H\underline{\phi}_1 + \underline{\tilde{z}}}$$

White term!!

Sufficient statistic

Because the whiteness of $\tilde{\underline{z}}$, the optimal detector is now simpler:

$$\hat{x}_1 = \lambda^* \underline{\phi}_1^H \underline{\underline{Q}}\,\underline{\underline{\Lambda}}^{-1/2} \tilde{\underline{y}} = \lambda^* \underline{\phi}_1^H \underbrace{\underline{\underline{Q}}\,\underline{\underline{\Lambda}}^{-1/2}\,\underline{\underline{\Lambda}}^{-1/2}\,\underline{\underline{Q}}^H}_{\underline{\underline{\Sigma}}^{-1}} \underline{y} = \lambda^* \underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{y}$$

that is:

$$\hat{x}_1 = \lambda^* \left[\underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1 \sqrt{p_1}\, x_1 + \underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{z} \right] \equiv \lambda^* \nu$$

with ‘λ’ such that:

$$\min_{\lambda} E\left[\left| x_1 - \hat{x}_1 \right|^2 \right] = E\left[\left| x_1 - \lambda^* \nu \right|^2 \right] \equiv E\left[(x_1 - \lambda^* \nu)^* \nu \right] = 0$$

or:

$$\lambda = \frac{E\left[x_1^* \nu \right]}{E\left[\left| \nu \right|^2 \right]}$$

One can see that:

$$1.-\quad E\left[x_1^{*}\nu\right] = \sqrt{p_1}\,\underline{\phi}_1^{H}\,\underline{\underline{\Sigma}}^{-1}\,\underline{\phi}_1\,\cancelto{1}{E\left[\left|x_1\right|^2\right]}$$

$$2.-\quad E\left[\left|\nu\right|^2\right] = p_1\left(\underline{\phi}_1^{H}\,\underline{\underline{\Sigma}}^{-1}\,\underline{\phi}_1\right)^2 + \underline{\phi}_1^{H}\,\cancel{\underline{\underline{\Sigma}}^{-1}}\,\cancel{\underline{\underline{\Sigma}}}\,\underline{\underline{\Sigma}}^{-1}\,\underline{\phi}_1$$

or:

$$\lambda = \frac{\sqrt{p_1}\left(\underline{\phi}_1^{H}\,\underline{\underline{\Sigma}}^{-1}\,\underline{\phi}_1\right)}{p_1\left(\underline{\phi}_1^{H}\,\underline{\underline{\Sigma}}^{-1}\,\underline{\phi}_1\right)^2 + \left(\underline{\phi}_1^{H}\,\underline{\underline{\Sigma}}^{-1}\,\underline{\phi}_1\right)} = \frac{\sqrt{p_1}}{1 + p_1\,\underline{\phi}_1^{H}\,\underline{\underline{\Sigma}}^{-1}\,\underline{\phi}_1}$$

that is:

$$\boxed{\begin{aligned} \hat{x}_1^{MMSE} &= \frac{\sqrt{p_1}}{1 + p_1\,\underline{\phi}_1^{H}\,\underline{\underline{\Sigma}}^{-1}\,\underline{\phi}_1}\,\underline{\phi}_1^{H}\,\underline{\underline{\Sigma}}^{-1}\,\underline{y} \\ \underline{\underline{\Sigma}} &= \underline{\underline{\phi}}_1^{H}\,\underline{\underline{P}}_1\,\underline{\underline{\phi}}_1 + \sigma^2\,\underline{\underline{I}} \end{aligned}}$$

We also see that:

$$\hat{x}_1 = \lambda^* \nu$$

$$E\left[\left|x_1 - \hat{x}_1\right|^2\right] = E\left[\left|x_1\right|^2\right] - E\left[x_1^{\,*}\hat{x}_1\right] - E\left[\hat{x}_1^{\,*}x_1\right] + E\left[\left|\hat{x}_1\right|^2\right] =$$

$$= E\left[\left|x_1\right|^2\right] - \lambda^* E\left[x_1^{\,*}\nu\right] - \lambda E\left[\nu^* x_1\right] + \left|\lambda\right|^2 E\left[\left|\nu\right|^2\right] \geq$$

$$\geq E\left[\left|x_1\right|^2\right] - \frac{\left|E\left[x_1^{\,*}\nu\right]\right|^2}{E\left[\left|\nu\right|^2\right]} = E\left[\left|x_1\right|^2\right] - \frac{p_1(\underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1)^2}{p_1(\underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1)^2 + \underline{\phi}_1 \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1} = 1 - \frac{p_1 \underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1}{1 + p_1 \underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1}$$

$$\lambda_{opt} = \frac{E\left[x_1 * \nu\right]}{E\left[\left|\nu\right|^2\right]}$$

or:

$$\boxed{\min E\left[\left|x_1 - \hat{x}_1\right|^2\right] = 1 - \frac{1}{1 + \frac{1}{SINR}} = \frac{1}{1 + SINR}}$$

We can conclude that:

1.- $\min E\left[\left|x_1 - \hat{x}_1\right|^2\right] \equiv \max SINR = \dfrac{S}{MAI + N}$

2.- The maximum SINR is given by:

$$\boxed{\max(\mathrm{SINR}) = \mathrm{p}_1 \underline{\phi}_1^{\mathrm{H}} \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1}$$

for: $\underline{\underline{\Sigma}} = E\left[\underline{z}\,\underline{z}^H\right] = \underline{\underline{S}}_1 \underline{\underline{P}}_1 \underline{\underline{S}}_1^H + \sigma^2 \underline{\underline{I}}$

(other formualtions .../...)

The common formulation becomes:

$$\min_{\underline{\omega}} E\left[\left|x_1 - \hat{x}_1\right|^2\right] = E\left[\left|x_1 - \underline{\omega}_1^H \underline{y}\right|^2\right] \equiv E\left[(x_1 - \underline{\omega}_1^H \underline{y})^* \underline{y}\right] = \underline{0}$$

or:

$$\boxed{E\left[x_1^* \underline{y}\right] = E\left[\underline{y}\,\underline{y}^H\right] \underline{\omega}_1}$$

where:

1.- $E\left[x_1^* \underline{y}\right] = \sqrt{p_1}\,\underline{\phi}_1$

2.- $E\left[\underline{y}\,\underline{y}^H\right] = p_1 \underline{\phi}_1 \underline{\phi}_1^H + \underline{\underline{\Sigma}}$

and we can make use of the '*Matrix Inversion Lemma*':

$$(\underline{\underline{A}} + \underline{\underline{U}}\,\underline{\underline{B}}\,\underline{\underline{V}})^{-1} = \underline{\underline{A}}^{-1} - \underline{\underline{A}}^{-1}\underline{\underline{U}}(\underline{\underline{B}}^{-1} + \underline{\underline{V}}\,\underline{\underline{A}}^{-1}\underline{\underline{U}})^{-1}\underline{\underline{V}}\,\underline{\underline{A}}^{-1}$$

for $\underline{\underline{\mathrm{A}}},\underline{\underline{\mathrm{B}}}$ non-singular

It is also interesting to see that the direct optimization of the *SINR* leads to the same result:

$$SINR = \frac{S}{MAI + N} = \frac{p_1 \left| \underline{\omega}_1^H \underline{\phi}_1 \right|^2}{E\left[\left| \underline{\omega}_1^H \underline{z} \right|^2 \right]} = p_1 \frac{\underline{\omega}_1^H \underline{\phi}_1 \underline{\phi}_1^H \underline{\omega}_1}{\underline{\omega}_1^H \underline{\underline{\Sigma}} \underline{\omega}_1} \qquad \text{(Rayliegh's quotient)}$$

$$\max \frac{\underline{\omega}_1^H \underline{\phi}_1 \underline{\phi}_1^H \underline{\omega}_1}{\underline{\omega}_1^H \underline{\underline{\Sigma}} \underline{\omega}_1} \quad \equiv \quad \underline{\phi}_1 \underline{\phi}_1^H \underline{\omega}_1 = \lambda' \underline{\underline{\Sigma}} \underline{\omega}_1$$

We found that: $\hat{x}_1 = \underline{\omega}_1^H \underline{y} = \lambda^* \underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{y} \Rightarrow \underline{\omega}_1 = \lambda \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1$

Thus: $\cancel{\lambda} \underline{\phi}_1 (\underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1) = \lambda' \cancel{\underline{\underline{\Sigma}}} \cancel{\underline{\underline{\Sigma}}}^{-1} \underline{\phi}_1 \cancel{\lambda} \Rightarrow \lambda' = (\underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1)!!$

And we also see that: $\max \lambda' = (\underline{\phi}_1^H \underline{\underline{\Sigma}}^{-1} \underline{\phi}_1) \;\; \alpha \;\; SINR!!$