# A novel estimator and theoretical limits for in-car radio-location

Department of Signal Theory and Communications. Universitat Politècnica de Catalunya.
Campus Nord, Ed. D5, Jordi Girona 1 i 3, 08034, Barcelona (Spain).
Email: {andreu,jriba}@gps.tsc.upc.es

Andreu Urruela and Jaume Riba

***Abstract*— This paper is focused on the design of a novel position estimator for in-car applications when some navigation information, such as speed and direction of the movement, can be supplied by additional integrated systems. Concretely, this paper presents a new method based on Time-of -Arrival (TOA) measurements exploiting the known, or partially known, drift speed of the mobile. The proposed algorithm transforms the TOA measurements to solve the classic non-linear problem presented in all TOA-based radio-location estimators, obtaining a linear model where the WLS approach can be applied. The associated lower bounds for the known or partially known speed cases are also computed in the paper to conclude that the proposed algorithm attains them asymptotically with large enough observation windows.**

## I. INTRODUCTION

A problem of growing importance in mobile communication networks is estimating the mobile position since the international commissions announced the minimum requirements in location for the next wireless generations and due to the recent interest for the location-based services. One of the most interesting application in the field of location-based services is the car-routing application where several navigation systems can assist the mobile positioning system with information such as inertial navigation systems (INS), speed of the mobile or direction of the mobile (compass).

The most well-known method for the position estimation in wireless systems is performed by measuring the time-of-arrival (TOA) or time-difference-of-arrival (TDOA) measurements amongst a set of references (normally BTSs) [1],[2]. The major problem of these kind of methods is the non-linear relationship between measurements and the parameters of interest or the difficult problem of finding the intersection of circles or hyperbolas. Several approaches for linearizing the equation system or simplifying the intersection problem have been presented, as in [3], where the problem was transformed into a linear intersection equation. The performance of methods based on TOA or TDOAs has been widely studied in recent publications as in [4] where the mobile was always assumed to be placed in a fixed position.

This article presents a new method based on TOA measurements that exploits the fact that the mobile is moving with a known or partially known drift speed, giving a new solution to the work presented in [5]. The associated lower bounds for this scenario are presented in order to compare the performances of the proposed estimator with them. Numerical simulations show that the proposed algorithm attains the theoretical limits under certain conditions. The new technique presented opens the door to future low complexity algorithms that exploit a partial or full knowledge of the mobile movement based on the available navigation information (speed, direction, etc).

## II. PROBLEM DEFINITION

The problem that we want to solve is finding a simple algorithm to estimate the mobile position when $K$ TOAs measurements are obtained at a constant rate for each base station (BTS) when $N$ different BTSs are available assuming that the mobile is moving in a linear trajectory and at a constant speed. Two different scenarios have been considered:

1) The speed of the mobile (this is the modulus and the angle of the speed) is perfectly known.
2) Only the modulus of the speed is provided.

First, lets us define the TOA measurement taken at the $k$-th instant of time using the $n$-th BTS:

$$t_{k,n} = f_{k,n}(\mathbf{x}, \mathbf{s}) + w_{k,n} = \|\mathbf{x} + k\mathbf{s} - \mathbf{z}_n\| + w_{k,n} \qquad (1)$$

where $1 \leq n \leq N$, $k$ is the temporal subindex inside the observation window of length $K$, $\mathbf{x}$ is the position of the mobile to be estimated, $\mathbf{z}_n$ is the position of the $n$-th BTS and $\mathbf{s}$ is the constant speed vector of the mobile (the constant speed assumption is common in in-car applications). Note that since the TOA measurements are obtained with a uniform sampling, the actual speed vector $\mathbf{s}_r$ in m/s is related with the $\mathbf{s}$ vector as $\mathbf{s}_r = \mathbf{s} \cdot r$ where $r$ is the sampling rate.

As in previous papers ([4]), $w_{k,n}$ is assumed gaussian with a known variance and stationary in time:

$$\mathrm{E}\left[w_{k,n} w_{k',n'}\right] = \delta_{n,n'} \delta_{k,k'} \sigma_n^2, \forall k, k'. \qquad (2)$$

where $\delta_{n,n'}$ is the Dirac-delta process. In equation (1), there are two possible definitions of the position of the mobile ($\mathbf{x}$). If we are interested in getting the position of the mobile in the middle of the observation window, then $-\frac{K-1}{2} \leq k \leq \frac{K-1}{2}$. On the other hand, if $\mathbf{x}$ is defined as the position at the end of the observation window, then $-(K-1) \leq k \leq 0$. Both problems can be generically formulated with (1).

## III. SPEED-AIDED ESTIMATOR

This section presents the main theoretical basis of the proposed estimator based on TOA measurements model shown in (1) and aided with the speed information (modulus and angle). The development of this algorithm can be explained in two steps: first an individual position estimate for each BTS is performed and second, a final estimate is obtained combining the partial ones.

Let us now develop the partial position estimate for the $n$-th BTS using only its $K$ TOA measurements. Equation (1) shows the clear non-linear relationship between measurements ($t_{k,n}$) and parameters of interest ($\mathbf{x}$). The proposed estimator is based on the model obtained with the squared TOA measurements. From (1) we have that the $K-th$ squared TOA of the $n$-th BTS can be expressed as follows:

$$t_{k,n}^2 = \|\mathbf{x} + k\mathbf{s} - \mathbf{z}_n\|^2 + w_{k,n}^2 + 2f_{k,n}(\mathbf{x},\mathbf{s}) \cdot w_{k,n} \quad (3)$$

In fact, $w_{k,n}^2$ is a random process with mean $\sigma_k^2$ so, we can split the two noise terms of (3) into a constant known term $\sigma_k^2$ and a zero-mean term $w'_{k,n} = {w_{k,n}}^2 + 2f_{k,n}(\mathbf{x},\mathbf{s}) \cdot w_{k,n} - \sigma_k^2$. Using this notation, and after straightforward operation from (3) we define our measurements model as:

$$t'_{k,n} \doteq t_{k,n}^2 - \sigma_k^2 - k^2 \|\mathbf{s}\|^2 = 2k \langle \mathbf{d}_n, \mathbf{s} \rangle + \|\mathbf{d}_n\|^2 + w'_{k,n} \quad (4)$$

where $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ stands for the inner product between two vectors and $\mathbf{d}_n = \mathbf{x} - \mathbf{z}_n$ is the vector from the $n$-th BTS to the mobile position to be estimated. Note that this position ($\mathbf{x}$) can be the position in the middle of the interval or the position at the end of the interval. Note also in (4) that the known speed modulus has been used to transform the TOA measurements. The variances of the transformed measurements in (4) can be expressed as follows:

$$\mathrm{E}\left[w'_{k,n} w'_{k',n'}\right] = \left(2\sigma_n^4 + 4f_{k,n}^2(\mathbf{x},\mathbf{s})\sigma_n^2\right) \delta_{n,n'} \delta_{k,k'} \quad (5)$$

$$\mathrm{E}\left[w'_{k,n} w'_{k',n'}\right] \simeq \left(4t_{k,n}^2 \sigma_n^2\right) \delta_{n,n'} \delta_{k,k'} \quad (6)$$

where this last simplification can be performed assuming low values for $\sigma_n$ ($\sigma_n << f_{k,n}(\mathbf{x},\mathbf{s}) \approx t_{k,n}$).

If now we stack the $K$ transformed measurements ($t'_{k,n}$) associated to the $n$-th BTS in a vector, we can write equation (4) in matrix form as:

$$\mathbf{t}'_n = \mathbf{A} \begin{bmatrix} 2\langle \mathbf{d}_n, \mathbf{s} \rangle \\ \|\mathbf{d}_n\|^2 \end{bmatrix} + \mathbf{w}'_n = \mathbf{A} \begin{bmatrix} f_{1,n} \\ f_{2,n} \end{bmatrix} + \mathbf{w}'_n \quad (7)$$

where $f_{1,n}$ and $f_{2,n}$ are defined in the trivial way and

$$\mathbf{A}_m = \begin{pmatrix} (K-1)/2 & 1 \\ \vdots & \vdots \\ -(K-1)/2 & 1 \end{pmatrix} \quad \mathbf{A}_e = \begin{pmatrix} 0 & 1 \\ \vdots & \vdots \\ -(K-1) & 1 \end{pmatrix} \quad (8)$$

are the two matrices associated to the case of being interested in the position in the middle ($\mathbf{A} = \mathbf{A}_m$) or at the end of the observation window ($\mathbf{A} = \mathbf{A}_e$).

Expression (7) shows us that the new transformed measurements ($\mathbf{t}'_n$) follow a linear evolution w.r.t. the position parameters ($f_{1,n}$ and $f_{2,n}$) and are corrupted by zero-mean independent noise terms. This linear relationship allows the proposed algorithm to extract explicitly the parameters of the model (related with the position of the mobile) without any kind of initialization or iterative method.

The proposed algorithm is, in fact, based on the application of the well-known weighted Least Square (WLS) principle to (7) to estimate the model parameters ($f_{1,n}$ and $f_{2,n}$) and, after several geometrical interpretations, compute the position of the mobile in a straightforward manner. As it is well known, the WLS algorithm is optimum in the case of gaussian additive noise. Approximation shown in (6) points out that the noise terms of the linear model ($\mathbf{w}'_n$) tends to be Gaussian for low values of $\sigma_k^2$. In the case of higher values of the noise variance $\sigma_k^2$, the WLS can not be considered optimum.

From (7), the WLS estimate of $f_{1,n}$ and $f_{2,n}$ can be expressed as follows:

$$\begin{bmatrix} \widehat{f}_{1,n} \\ \widehat{f}_{2,n} \end{bmatrix} = \left(\mathbf{A}^T \widehat{\mathbf{R}}_{w'_n}^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^T \widehat{\mathbf{R}}_{w'_n}^{-1} \mathbf{t}'_n \quad (9)$$

where $\widehat{\mathbf{R}}_{w'_n}$ is a diagonal matrix formed with the approximations shown in (6):

$$\left[\widehat{\mathbf{R}}_{w'_n}\right]_{k,k'} = \left(4t_{k,n}^2 \sigma_n^2\right) \delta_{k,k'} \quad (10)$$

where $[\mathbf{M}]_{k,k'}$ is the $k$-th row, $k'$-th column element of a matrix $\mathbf{M}$.

Figure 1 shows us two important parameters related with the position and the speed of the mobile. These parameters are $d_{1,n}$ and $d_{2,n}$. The first one corresponds to the perpendicular distance from the $n$-th BTS to the trajectory line. The second one is the distance from the actual position of the mobile to the intersection between the trajectory line and the perpendicular that pass through the BTS position. It is not difficult to see that the second parameter ($d_{2,n}$) is the projection of $\mathbf{d}_n$ to the mobile trajectory line defined by the normalized speed. With these definitions, we can write:

$$\|\mathbf{d}_n\|^2 = d_{1,n}^2 + d_{2,n}^2 \quad (11)$$

$$d_{2,n} = \frac{\langle \mathbf{d}_n, \mathbf{s} \rangle}{\|\mathbf{s}\|} \quad (12)$$

Now, applying (7), (11) and (12) we found an estimate of the geometrical parameters $d_{1,n}$ and $d_{2,n}$ shown in figure (1) based on the WLS estimates $\widehat{f}_{1,n}$ and $\widehat{f}_{2,n}$ shown in (9):

$$\widehat{d}_{1,n} = \sqrt{\widehat{f}_{2,n} - \widehat{d}_{2,n}^2} \quad (13)$$

$$\widehat{d}_{2,n} = \widehat{f}_{1,n}/2\|\mathbf{s}\| \quad (14)$$

Note that, the developed expressions until now do not require the knowledge of the angle of the speed (only the modulus has been used). Now, if the angle is provided, the position estimate for the $n$-th BTS can be computed as follows (See figure 1):

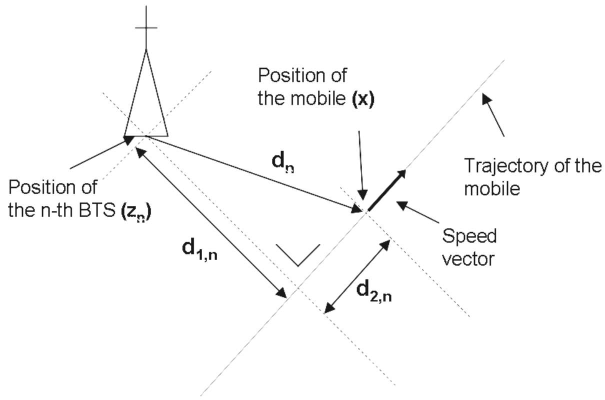


Fig. 1. Graphical parameters of the mobile position

$$\widehat{\mathbf{x}}_n = \mathbf{z}_n - \widehat{d}_{1,n}\overline{\mathbf{s}}^{\perp} - \widehat{d}_{2,n}\overline{\mathbf{s}} \quad (15)$$

where $\overline{\mathbf{s}}$ and $\overline{\mathbf{s}}^{\perp}$ are the unitary vectors in the direction of the speed and in the perpendicular respectively. Note that there is a joint ambiguity in the definition of $\overline{\mathbf{s}}^{\perp}$ in (15) and $\widehat{d}_{1,n}$ in (13) that has to be solved with several BTS's or with the additional information as i.e. the sectorization of the BTS's.

Finally, the final position estimate combining the $N$ partial estimates obtained with (15) is performed as follows:

$$\widehat{\mathbf{x}} = \left[\sum_{n'} \widehat{\mathbf{R}}_{n'}^{-1}\right]^{-1} \sum_{n} \widehat{\mathbf{R}}_n^{-1}\widehat{\mathbf{x}}_n \quad (16)$$

where $\widehat{\mathbf{R}}_n$ is an estimate of the covariance matrix of $\widehat{\mathbf{x}}_n$ ($\mathbf{R}_n = \mathrm{E}\left[\widehat{\mathbf{x}}_n - \mathbf{x}\right]\left[\widehat{\mathbf{x}}_n - \mathbf{x}\right]^T$) and it is obtained as follows:

$$\widehat{\mathbf{R}}_n^{-1} = \sigma_n^{-2} \sum_k \mathbf{v}_{k,n}\mathbf{v}_{k,n}^T \quad (17)$$

where $\mathbf{v}_{k,n}$ is the unitary vector from the $n$-th BTS to the position estimated related with $k$-th TOA measurement: $\mathbf{v}_{k,n} = \widehat{\mathbf{x}}_n + k\mathbf{s} - \mathbf{z}_n / \left\|\widehat{\mathbf{x}}_n + k\mathbf{s} - \mathbf{z}_n\right\|$;

The estimate of $\mathbf{R}_n$ shown in (17) is, in fact, the approximate CRB expression of the position estimation using only the $n$-th BTS. As $\lim_{K\to\infty} \widehat{\mathbf{x}}_n = \mathbf{x}$, the estimate $\widehat{\mathbf{R}}_n$ tends to be the theoretical limit of the accuracy using only the n-th BTS, this is the exact covariance matrix of $\widehat{\mathbf{x}}_n$. Then, equation (16) performs optimum combination with large enough observation window ($K \to \infty$).

## IV. MODULUS SPEED-AIDED ESTIMATOR

The algorithm proposed in the previous section assumed that both the modulus and the angle of the speed were completely known. This section is focused on developing a position estimator that only assumes that the modulus of the speed is known which may be a possible practical situation.

The development of the previous section can be exactly repeated in this case except the final step performed in (15). This step can not be performed because $\overline{\mathbf{s}}$ and $\overline{\mathbf{s}}^{\perp}$ were formed with the a priori known angle of speed ($\alpha_{\mathbf{s}}$).

The solution presented in this section consists in developing a consistent closed-form estimator of the angle of the speed using the same parameters $\widehat{d}_{1,n}$ and $\widehat{d}_{2,n}$ obtained in (15) involving all available BTSs. Once $\widehat{\alpha}_{\mathbf{s}}$ is obtained, the same equation (15) is used to compute the partial position estimates and (16) is used to combine them.

The idea of the closed-form $\alpha_{\mathbf{s}}$ estimator is based on the observation that using (15) there are $N$ partial estimates of the position that depend on the angle of the speed: $\widehat{\mathbf{x}}_n(\alpha_{\mathbf{s}})$.

The idea here is finding the angle of the speed that minimize the mean dispersion of the position estimates:

$$\Phi(\alpha_{\mathbf{s}}) = \sum_n \left\| \widehat{\mathbf{x}}_n(\alpha_{\mathbf{s}}) - \frac{1}{N}\sum_{n'} \widehat{\mathbf{x}}_{n'}(\alpha_{\mathbf{s}}) \right\| \quad (18)$$

being $\widehat{\alpha}_{\mathbf{s}}$ the minimizer of $\Phi(\alpha_{\mathbf{s}})$. Now, from (15) and (18), we have:

$$\Phi(\alpha_{\mathbf{s}}) = C_t - 2\sum_n \Delta\mathbf{z}_n^T \left[\Delta\widehat{d}_{1,n}\overline{\mathbf{s}}^{\perp}(\alpha_{\mathbf{s}}) + \Delta\widehat{d}_{2,n}\overline{\mathbf{s}}(\alpha_{\mathbf{s}})\right] \quad (19)$$

where

$$\Delta\mathbf{z}_n = \mathbf{z}_n - \frac{1}{N}\sum_{n'} \mathbf{z}_{n'} \quad (20)$$

$$\Delta\widehat{d}_{1,n} = \widehat{d}_{1,n} - \frac{1}{N}\sum_{n'} \widehat{d}_{1,n'} \quad (21)$$

$$\Delta\widehat{d}_{2,n} = \widehat{d}_{2,n} - \frac{1}{N}\sum_{n'} \widehat{d}_{2,n'} \quad (22)$$

$$\overline{\mathbf{s}}(\alpha_{\mathbf{s}}) = \begin{bmatrix} \sin(\alpha_{\mathbf{s}}) \\ \cos(\alpha_{\mathbf{s}}) \end{bmatrix} \quad (23)$$

$$\overline{\mathbf{s}}^{\perp}(\alpha_{\mathbf{s}}) = \begin{bmatrix} -\cos(\alpha_{\mathbf{s}}) \\ \sin(\alpha_{\mathbf{s}}) \end{bmatrix} \quad (24)$$

and $C_t$ is an irrelevant constant. Differentiating the cost function shown in (19) with respect $\alpha_{\mathbf{s}}$ for the particular case of 2D (localization problem in a 2 Dimensional plane), we found:

$$\frac{\partial}{\partial\alpha_{\mathbf{s}}}\Phi(\alpha_{\mathbf{s}}) = -2\sum_n \Delta\mathbf{z}_n^T \left[-\Delta\widehat{d}_{1,n}\overline{\mathbf{s}}(\alpha_{\mathbf{s}}) + \Delta\widehat{d}_{2,n}\overline{\mathbf{s}}^{\perp}(\alpha_{\mathbf{s}})\right] \quad (25)$$

Note from (23) and (24), the time derivatives of the unitary speed vectors can be computed as follows:

$$\frac{\partial\overline{\mathbf{s}}(\alpha_{\mathbf{s}})}{\partial\alpha_{\mathbf{s}}} = \overline{\mathbf{s}}^{\perp}(\alpha_{\mathbf{s}})$$

$$\frac{\partial\overline{\mathbf{s}}^{\perp}(\alpha_{\mathbf{s}})}{\partial\alpha_{\mathbf{s}}} = -\overline{\mathbf{s}}(\alpha_{\mathbf{s}})$$

Finally, solving $\frac{\partial}{\partial\alpha_{\mathbf{s}}}\Phi(\alpha_{\mathbf{s}}) = \mathbf{0}$ we found the closed-form estimator:

$$\widehat{\alpha}_{\mathbf{s}} = \arctan\frac{\sum_n \left[\Delta\mathbf{z}_n\right]_1 \Delta\widehat{d}_{1,n} + \left[\Delta\mathbf{z}_n\right]_2 \Delta\widehat{d}_{2,n}}{\sum_n \left[\Delta\mathbf{z}_n\right]_1 \Delta\widehat{d}_{2,n} - \left[\Delta\mathbf{z}_n\right]_2 \Delta\widehat{d}_{1,n}} \quad (26)$$

where $\left[\Delta\mathbf{z}_n\right]_1$ and $\left[\Delta\mathbf{z}_n\right]_2$ are the first and second component of the vector $\Delta\mathbf{z}_n$ respectively.

## V. Theoretical limits

The Cramer-Rao lower bound (CRB) will be computed for the following cases:

- The speed of the mobile is completely known. ($\mathbf{CRB}_{SK}$)
- The modulus of the mobile speed is known. ($\mathbf{CRB}_{MK}$)
- No information about the speed is provided ($\mathbf{CRB}_{NS}$).

These three theoretical limits will allow us to evaluate the performances of the proposed estimators. First, let us define the joint pdf distribution of the available data presented in (1) following the noise variance definition shown in (2):

$$p(\mathbf{t}|\mathbf{s}) = \prod_{k,n} \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{1}{2}\frac{\left(t_{k,n}-f_{k,n}(\mathbf{x},\mathbf{s})\right)^2}{\sigma_n^2}} \tag{27}$$

where $\mathbf{t}$ is formed by $K \cdot N$ TOA measurements ($t_{k,n}$) stacked into a vector.

The general procedure to compute the three limits consists in computing the CRB of the position coordinates treating the unknown speed parameters (angle and/or modulus) as nuisance parameters [6]. To this end, we need to first compute the CRB for all unknown parameters, and latter extract the submatrix associated only with the position parameters. The expression of the three theoretical limits can be expressed as follows:

$$\mathbf{CRB}_{SK} = \mathbf{I}_{\mathbf{x},\mathbf{x}}^{-1} \tag{28}$$

$$\mathbf{CRB}_{MK} = \left(\mathbf{I}_{\mathbf{x},\mathbf{x}} - \frac{\mathbf{i}_{\mathbf{x},\alpha_{\mathbf{s}}}\mathbf{i}_{\mathbf{x},\alpha_{\mathbf{s}}}^T}{i_{\alpha_{\mathbf{s}},\alpha_{\mathbf{s}}}}\right)^{-1} \tag{29}$$

$$\mathbf{CRB}_{NS} = \left(\mathbf{I}_{\mathbf{x},\mathbf{x}} - \mathbf{I}_{\mathbf{x},\mathbf{s}}\mathbf{I}_{\mathbf{s},\mathbf{s}}^{-1}\mathbf{I}_{\mathbf{x},\mathbf{s}}^T\right)^{-1} \tag{30}$$

where $\alpha_{\mathbf{s}}$ is the angle of the speed vector $\mathbf{s}$ and the 2x2 matrices $\mathbf{I}_{\mathbf{x},\mathbf{x}}$, $\mathbf{I}_{\mathbf{x},\mathbf{s}}$, $\mathbf{I}_{\mathbf{s},\mathbf{s}}$, the 2x1 vector $\mathbf{i}_{\mathbf{x},\alpha_{\mathbf{s}}}$ and the scalar $i_{\alpha_{\mathbf{s}},\alpha_{\mathbf{s}}}$ are defined as follows:

$$[\mathbf{I}_{\mathbf{x},\mathbf{x}}]_{p,q} = -\mathrm{E}\left[\frac{\partial^2}{\partial \mathbf{x}_p \partial \mathbf{x}_q}\ln p(\mathbf{t}|\mathbf{s})\right] \tag{31}$$

$$[\mathbf{I}_{\mathbf{x},\mathbf{s}}]_{p,q} = -\mathrm{E}\left[\frac{\partial^2}{\partial \mathbf{x}_p \partial \mathbf{s}_q}\ln p(\mathbf{t}|\mathbf{s})\right] \tag{32}$$

$$[\mathbf{I}_{\mathbf{s},\mathbf{s}}]_{p,q} = -\mathrm{E}\left[\frac{\partial^2}{\partial \mathbf{s}_p \partial \mathbf{s}_q}\ln p(\mathbf{t}|\mathbf{s})\right] \tag{33}$$

$$[\mathbf{i}_{\mathbf{x},\alpha_{\mathbf{s}}}]_{p} = -\mathrm{E}\left[\frac{\partial^2}{\partial \mathbf{x}_p \partial \alpha_{\mathbf{s}}}\ln p(\mathbf{t}|\mathbf{s})\right] \tag{34}$$

$$i_{\alpha_{\mathbf{s}},\alpha_{\mathbf{s}}} = -\mathrm{E}\left[\frac{\partial^2}{\partial {\alpha_{\mathbf{s}}}^2}\ln p(\mathbf{t}|\mathbf{s})\right] \tag{35}$$

where $[\mathbf{M}]_{p,q}$ is the $p$-th row $q$-th collum element of a matrix $\mathbf{M}$, $[\mathbf{m}]_p$ is the $p$-th element of a vector $\mathbf{m}$ , and finally $\mathbf{x}_p$ and $\mathbf{s}_p$ are the $p$-th component of the position and speed respectively (p={1,2}). Following these definitions and after straightforward mathematical operations, the three mentioned theoretical limits for the 2D case are:

$$\mathbf{CRB}_{SK} = {\mathbf{D}_0}^{-1} \tag{36}$$

$$\mathbf{CRB}_{MK} = \left[\mathbf{D}_0 - \mathbf{D}_1\left[\frac{\left(\overline{\mathbf{s}}^{\perp}\right)\left(\overline{\mathbf{s}}^{\perp}\right)^T}{\left(\overline{\mathbf{s}}^{\perp}\right)^T\mathbf{D}_2\left(\overline{\mathbf{s}}^{\perp}\right)}\right]{\mathbf{D}_1}^T\right]^{-1} \tag{37}$$

$$\mathbf{CRB}_{NS} = \left[\mathbf{D}_0 - \mathbf{D}_1{\mathbf{D}_2}^{-1}{\mathbf{D}_1}^T\right]^{-1} \tag{38}$$

where

$$\mathbf{D}_i = \sum_k k^i \sum_n \sigma_n^{-2}\mathbf{u}_{k,n}\mathbf{u}_{k,n}^T \tag{39}$$

being $\mathbf{u}_{k,n}$ the unitary vector from the $n$-th BTS to the mobile position at the $k$-th instant of time:

$$\mathbf{u}_{k,n} = \frac{\mathbf{x} + k\mathbf{s} - \mathbf{z}_n}{\|\mathbf{x} + k\mathbf{s} - \mathbf{z}_n\|} \tag{40}$$

## VI. Simulations

Numerical simulations presented here compare the performances of the proposed algorithm with the theoretical limits as a function of the time observation window. These simulations are performed using four BTSs placed at [500,0] , [0,500],[-500,0] and [0,-500] in meters and the mobile position in the middle/end point of the trajectory is placed in the point [0,0]. The simulations assume that the TOA measurements of all BTSs have the same variance. Several values for these variances have been used in order to study the impact of high variances in the algorithm performance. All TOA measurements are always taken at a rate of 1 sample/second and the mobile is moving in a 45 degrees trajectory with an speed of 50 Km/h.

In the figure (2) we can see the m.s.e in the estimation of the position in the middle point of the observation window. It can be seen that the three theoretical limits considered are practically identical. This is due to the the symmetry in the problem geometry in a constant speed movement. This facts points out that the main advantage of the proposed algorithm in this scenario (estimation of the middle point position) is the availability due to the fact that, solving some simple ambiguities, it is possible to get the position using only one BTS when the minimum number of BTS using TOA measurement in a static scenario is two. Analogously to the CRBs, the performance of the algorithm assuming or not the knowledge of the angle of the speed are quite similar. It can also be observed that with low values in the length of the observation window ($K$), especially in high TOA variance scenarios, the performance of the algorithm is poorer than the CRB's. This is due to a deficient estimation of the individual error covariances matrices $\widehat{\mathbf{R}}_n$ in the combination process shown in (16). In general this effect will appear when the individual position estimates have high variances, this is, in short observation window and/or with higher TOA-variances. On the other hand, the algorithm attains asymptotically the theoretical limits for large enough observation windows.

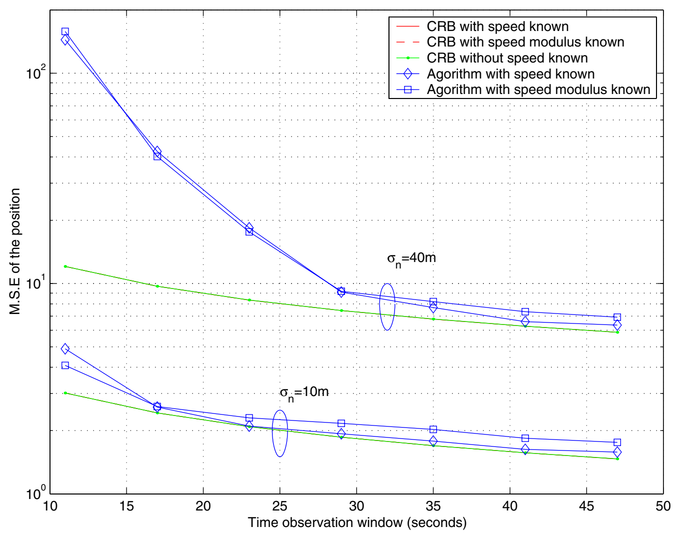


Fig. 2. Theoretical limits (CRBs) and the algorithm performances (estimation of the position in the mid-point)

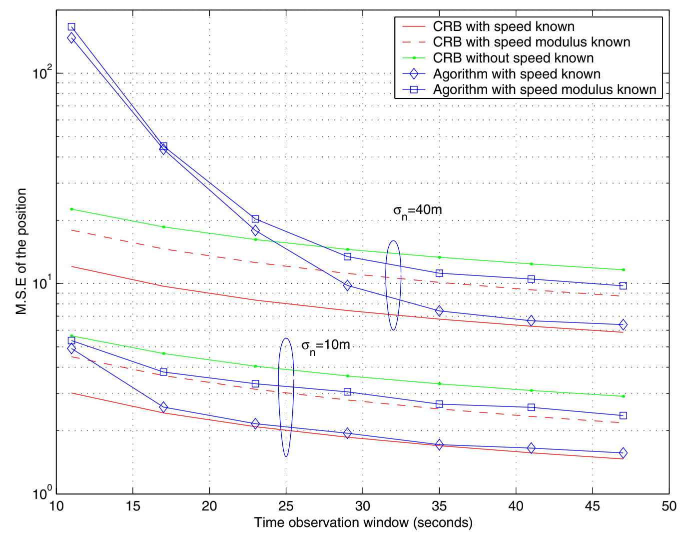


Fig. 3. Theoretical limits (CRBs) and the algorithm performances (estimation of the position in the end-point)

Figure (3) shows the same simulation but in the case of being interested in the position of the mobile at the end of the observation window. Maybe, this is a more realistic scenario in witch we are interested in the current position of the mobile. In this case, the symmetry of the problem is broken and the three CRB's differ. In this case, the knowledge of the modulus of the speed and its direction improves the performances of the algorithm. It can also be seen that the algorithm that exploits only the knowledge of the mobile speed modulus attains the associated CRB and the algorithm that exploits also the known angle of the speed attains the best CRB. In this case the advantage of the use of speed information is reflected not only in availability like in the previous case but also in accuracy. Finally, the same effect for short observation window and/or high TOA-variances appears, thus degrading the performances of the algorithm.

## VII. Conclusions

The proposed algorithm solves the classical non-linear relationship problem of all TOA-based position estimators by transforming the observed TOA measurements and obtaining a linear relationship between the observed transformed measurements and a set of parameters related with the position. After some assumptions about the transformed noise it is possible to extract the linear model parameters using the WLS approach. After several geometric interpretation, it is possible to find a simple algorithm that computes the position in a simple way using the extracted parameters. Two versions of the proposed algorithm have been presented in this paper. The first one, exploiting the complete knowledge of the speed, can compute an individual estimate of the position for each BTS involved in the problem and latter it combines all these individual estimates to compute a final one that attains asymptotically the associated theoretical limit. The second one, exploiting only the knowledge of the modulus of the speed, computes a consistent estimate of the angle of the speed using all the parameters extracted from all signals and latter it computes the final position estimate following a procedure similar to the first one. This second version of the algorithm also attains asymptotically its associated CRB.

The major advantages of this new algorithm are the low implementation complexity, the lack of initialization (very common in the classic Taylor series based location algorithms) and the improvements in accuracy and availability. The availability is improved because, it is possible to get the position using only one BTS if a complete knowledge of the speed is provided (a simple ambiguity has to be solved in this case with, for instance, the sectorization of the BTS). The availability is only improved significantly in the case of estimating the position of the mobile at the end of the time observation window.

## References


[1] Y.T. Chain and K. Ho. A simple and efficient estimator for hyperbolic location. In *IEEE transactions on signal Processing*, volume 42, pages pp. 1905–1915, Agust 1994.

[2] D.J Torrieri. Statistical theory of passive location sytems. In *IEEE transactions on Aerospace and Electronic Systems*, volume AES-20, pages pp. 183 – 198, March 1984.

[3] J. Jr. Caffery. A new approach to the geometry of TOA location. In *Vehicular Technology Conference, 2000. IEEE VTS-Fall VTC 2000. 52nd. Boston, Ma Usa, September 24-28, 2000*, volume 4.

[4] Maurizio A. Spirito. On the accuracy of cellular mobile station location. *IEEE Transactions on Vehicular Technology*, 50(3): pp. 674–685, May 2001.

[5] A. Urruela and J.Riba. A novel estimator and performance bound for time propagation delay and doppler based radio-location. In *ICASSP'03. 6-10 April 2003 Hong Kong*.

[6] Louis L. Scharf. *Statistical signal processing. Detection, estimation and time series analysis*. Addison Wesley, 1991.