Efficient mobile location from time measurements with unknown variances in dynamic scenarios

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Abstract — **This paper is focused on the study of the Maximum Likelihood (ML) mobile position estimator when the quality of the available measurements is not a-priori known. Based on a statistical analysis, a polynomial time-evolution model is used to simplify the ML function, finding a closed-form approximation of the ML estimator. Numerical simulations show that the proposed algorithm, with a low implementation complexity, attains the Cramer Rao Lower Bound (CRB) for all reasonable observed window lengths and for any arbitrary distribution of the measurement variances. Although the mathematical development of this closed-form position estimator is quite dense, the obtained algorithm has a very low complexity implementation.**

I Introduction

A problem of growing importance in mobile communication networks is finding the position of the mobile terminals. This will be mandatory for public-access networks and very useful for a great variety of position-based services.

The most common approach for this problem is computing the position of the mobile from a set of time measurements among a set of base stations (BSs). These time measurements may consist of Time of Arrival (TOA) or Time Difference of Arrival measurements (TDOA).

The quality of these measurements depends clearly on the distance from the mobile to the BS, so it can take very different values. Additionally, it is quite difficult to have an a-priori knowledge of the variances due to the fact that they also depend of the environment conditions. The knowledge of these variances is important in the location algorithms because it allows an optimal data fusion specially in scenarios where measurements with very different quality are present. The lack of knowledge of the variances of the available measurements may impact in significant position errors.

In the recent literature, there have been presented multiple location estimators based on time measurements as in [3],[4],[6] and the analysis of the theoretical limits (CRB) has also been presented in [5]. The previous position estimators presented in the literature assumed an a-priori knowledge of the quality of the available measurements.

This paper presents a deep study of the Maximum (ML) estimation of the mobile position in the case of Known and Unknown Variances using a common model for both cases. From this common model, a closed-form position estimator is derived. This estimator presents a low implementation complexity and attains the CRB for all the reasonable lengths of the observed window.

II Signal model

Let us assume that K independent TOA/TDOA measurements can be taken at each instant of time (from K BSs in the case of TOA measurements and from $K+1$ BSs in the case of TDOA measurements). Let us also assume that we take measurements at N instants of time at a constant rate r, so the n-th measurement (in time) of the k-th source can be expressed as follows:

$$t_{k,n} = f_{k,n}\left(\mathbf{x},\mathbf{u}\right) + w_{k,n} \quad k \in [1,K]\,,\ n \in [-(N-1),0] \tag{1}$$

where $f_{k,n}\left(\mathbf{x},\mathbf{u}\right)$ defines the relationship between the measurement ($t_{k,n}$) and the mobile position $\mathbf{x}$ (defined at the end of the observed temporal window) and the parameters of the mobile movement model $\mathbf{u}$. Note also that $w_{k,n}$ is commonly assumed as in [5] a white, zero-mean gaussian noise:

$$E\left[w_{k,n}w_{n',k'}\right] = \delta_{n-n'}\delta_{k-k'}\sigma_k^2. \tag{2}$$

where δ_n is the Kronecker delta. The definition of $f_{k,n}\left(\mathbf{x},\mathbf{u}\right)$in (1) depends on the specific type of measurements we have. In the case of TOA/TDOA measurements, it can be decomposed as follows:

$$f_{k,n}\left(\mathbf{x},\mathbf{u}\right) = g_k\left(\mathbf{h}_n\left(\mathbf{x},\mathbf{u}\right)\right) \tag{3}$$

where $g_k\left(\mathbf{y}\right)$ is a function that uniquely depends on the type of measurement (TOA/TDOA) and on the geometry of the problem (basically, the position of the BSs) and $\mathbf{h}_n\left(\mathbf{x},\mathbf{u}\right)$ is the function that models the mobile movement. This decomposition will be useful in the development of the closed-form ML estimator after developing the ML formulation. The definition of $g_k\left(\mathbf{y}\right)$ for TOA and TDOA measurements and an example of a simple model for $\mathbf{h}_n$ are presented here:

$$g_k^{TOA}\left(\mathbf{y}\right) = \left|\mathbf{y}-\mathbf{x}_k^{BTS}\right| \tag{4}$$

$$g_k^{TDOA}\left(\mathbf{y}\right) = \left|\mathbf{y}-\mathbf{x}_{k+1}^{BTS}\right| - \left|\mathbf{y}-\mathbf{x}_1^{BTS}\right| \tag{5}$$

$$\mathbf{h}_n\left(\mathbf{x},\mathbf{u}\right) = \mathbf{x} + n\frac{\mathbf{s}_0}{r} = \mathbf{x} + n\mathbf{s} \tag{6}$$

where $\mathbf{x}_m^{BTS}$ is the known position of the m-th BS, $\mathbf{s}_0$ is the speed in meters per second and $\mathbf{s} = \frac{\mathbf{s}_0}{r}$ is the unknown speed vector in meters per sample. In this approach it has been assumed a linear model evolution only considering the speed of the mobile (this is $\mathbf{u} = \mathbf{s}$) but more sophisticated models can be considered taking into account the acceleration or higher order movement derivatives. Note that as the range of n is defined as (1), $\mathbf{x}$ in (6) becomes the mobile position at the end of the observed window. The range of subindex n will be omitted sometimes for space reasons.

[1]This work has been partially supported by the European Commission under IST project EMILY IST-2000-26040 and by the following research projects of the Spanish/Catalan Science and Technology Commissions (CICYT/CIRIT): TIC2003-05482, TIC2002-04594, TIC2001-2356, TIC2000-1025 and 2001SGR-00268.

III Joint ML formulation

As we have $N \cdot K$ independent measurements, the ML estimation of all unknowns (in general) becomes:

$$\widehat{\mathbf{x}}, \widehat{\mathbf{u}}, \widehat{\sigma} = \arg\max_{\mathbf{x},\mathbf{u},\sigma} \Phi = \arg\max_{\mathbf{x},\mathbf{u},\sigma} \prod_{k=1}^{K} \prod_{n} p_{k,n} \left(t_{k,n} | \mathbf{x}, \mathbf{u} \right) \quad (7)$$

where Φ is the ML function, $p_{k,n} \left(t_{k,n} | \mathbf{x}, \mathbf{u} \right)$ is the individual p.d.f of the measurement $t_{k,n}$ normally distributed as $\mathcal{N}\left(f_{k,n} \left(\mathbf{x}, \mathbf{u} \right), \sigma_k^2 \right)$, $\sigma = [\sigma_1, \cdots, \sigma_K]$ is the vector of all unknown variances and finally $\widehat{\mathbf{x}}$, $\widehat{\mathbf{u}}$ and $\widehat{\sigma}$ are the ML estimates of their respective parameters.

Now, after applying the negative natural logarithm, we have:

$$\Phi_1 = -2 \ln \Phi = C_1 + N \sum_{k=1}^{K} \ln \sigma_k^2 + \sum_{n} \sum_{k=1}^{K} \frac{\left(t_{k,n} - f_{k,n} \left(\mathbf{x}, \mathbf{u} \right) \right)^2}{\sigma_k^2} \quad (8)$$

where through the paper, C_p will always refer to irrelevant constants ($\forall p$). In the particular case of completely known variances, the first term in Φ_1 is constant w.r.t the unknown parameters so it may be removed as follows:

$$\Phi_2^{KV} = C_2 + \sum_{n=-(N-1)}^{0} \sum_{k=1}^{K} \sigma_k^{-2} \left(t_{k,n} - f_{k,n} \left(\mathbf{x}, \mathbf{u} \right) \right)^2 \quad (9)$$

where KV stands for *Known Variances*.

IV ML function in the Unknown Variances case

In this section, the development of the previous section is addressed in the specific case of completely unknown variances. Three steps will be required to obtain expressions similar to the Known Variance case and to be able to formulate a general framework for both cases:

- Compress the ML estimation of the K unknown variances σ_k in the general ML function Φ_1 defined in (8).
- Approximate the time evolution of the measurements with a polynomial model.
- Select concrete properties of the polynomial model in order to simplify the expression of the ML function.

These three steps are shown in the following three subsections

IV-A Compressed ML function

From (8), it can be computed the individual ML estimates of the K variances σ_k assuming that $\mathbf{x}$ and $\mathbf{u}$ are known. These are defined as $\widehat{\sigma_k^2} = \arg\max_{\sigma_k^2} \Phi_1$. Taking the definition of Φ_1 in (8), they can be expressed as:

$$\widehat{\sigma_k^2} = \frac{1}{N} \sum_{n=-(N-1)}^{0} \left(t_{k,n} - f_{k,n} \left(\mathbf{x}, \mathbf{u} \right) \right)^2 \quad (10)$$

Using this expression in (8), we obtain the compressed-ML function as:

$$\Phi_2^{UV} = \Phi_1 \Big|_{\sigma_k^2 = \widehat{\sigma_k^2}} = C_3 + N \sum_{k=1}^{K} \ln \left[\frac{1}{N} \sum_{n} \left(t_{k,n} - f_{k,n} \left(\mathbf{x}, \mathbf{u} \right) \right)^2 \right] \quad (11)$$

where UV stands for *Unknown Variances*. This approach is commonly referred to as Conditional Maximum Likelihood (CML) [1]. Actually, this work is partially motivated by the similarity with the compressed covariance matrix approach in Direction of Arrival (DOA) estimation in the array field.

IV-B Time evolution approximation

First of all, a linear approximation is used to restrict the time evolution of the TOA/TDOA measurements to a linear combination of P time-evolution fixed vectors (time basis). This can be mathematically expressed as follows:

$$f_{k,n} \left(\mathbf{x}, \mathbf{u} \right) \approx \sum_{p=0}^{P-1} w_p \left(n \right) \gamma_{k,p} \left(\mathbf{x}, \mathbf{u} \right) \quad (12)$$

where $\gamma_{k,p} \left(\mathbf{x}, \mathbf{u} \right)$ are the coefficients (coordinates of the time basis in algebraic language) that best fit the model and $w_p \left(n \right)$ are the time basis used. It is important to note that the P functions $\gamma_{k,p} \left(\mathbf{x}, \mathbf{u} \right)$ (associated to the k-th source) depend only on the parameters of interest (the mobile position and the movement parameters) but they do not depend on the time subindex n. On the other hand, the P time windows $w_p \left(n \right)$ do not dependent on the parameter of interest but they only describe the P a-priori fixed time-evolution models (time basis).

The use of this linear approximation is clearly motivated by a statistical study of the time-evolution of TOA/TDOA measurements. Concretely, it has been obtained the singular value decomposition (SVD) of the time evolution of TOA/TDOA measurements in a constant speed scenario for a reasonable range of speeds and for a variety of geometries (position of the BSs). Although the results of these simulations have not been included in this paper for space reasons, the results conclude that the first P eigenvalues practically dominate over the rest of eigenvalues. This result justify the approximation made in (12) taking as time basis ($w_p \left(n \right)$) the P eigenvectors associated with the P higher eigenvalues.

On the other hand looking at the these P eigenvectors associated with the dominant P eigenvalues, it can be observed that they practically satisfy a polynomial evolution as:

$$\widetilde{w}_p \left(n \right) = n^p \quad (13)$$

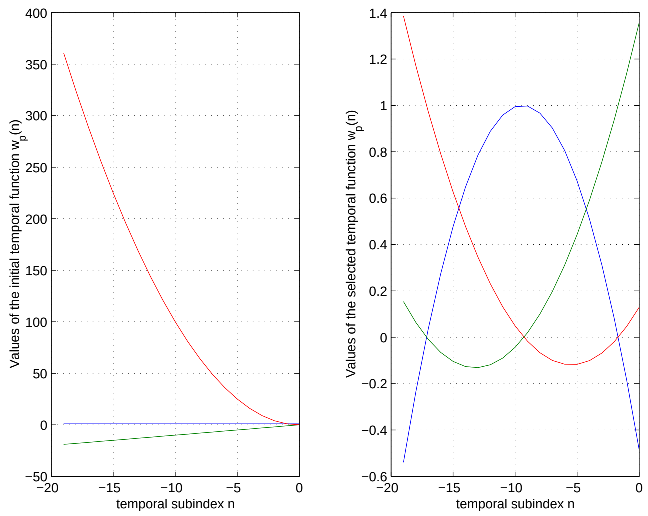

Figure 1: Vectors of the time basis

This initial time basis, extracted from numerical simulations, is depicted for $P = 3$ and $N = 20$ in the left side of figure (1)). Finally, this initial time basis obtained by simulation is not the unique possibility because we can choose as base any linear combination of this initial basis as (base change):

$$\mathbf{W} = \widetilde{\mathbf{W}} \cdot \mathbf{Q} \tag{14}$$

where the element of the n-th row and p-th column of $\mathbf{W}$ and $\widetilde{\mathbf{W}}$ are $w_p(n)$ and $\widetilde{w}_p(n)$ respectively and the matrix $\mathbf{Q}$ is any non-singular PxP matrix. The right side of figure (1) shows an example of the P vectors of the time basis obtained with a particular matrix $\mathbf{Q}$ satisfying some constraints explained later.

From (14), we can formulate the p-th window model as:

$$w_p(n) = \sum_{i=0}^{P-1} [\mathbf{Q}]_{i,p} \, \widetilde{w}_i(n) = \sum_{i=0}^{P-1} [\mathbf{Q}]_{i,p} \, n^i \tag{15}$$

where $[\mathbf{Q}]_{i,p}$ is the i-th row and p-th column element of matrix $\mathbf{Q}$.

IV-C Polynomial basis selection

Taking (11) and (12), we can write the ML function Φ_2^{UV} as:

$$\Phi_2^{UV} = C_3 + N \sum_{k=1}^{K} \ln \left[\frac{1}{N} \sum_n \left(t_{k,n} - \sum_{p=0}^{P-1} w_p(n) \, \gamma_{k,p}(\mathbf{x},\mathbf{u}) \right)^2 \right] \tag{16}$$

It will become clear later on that a good property for the selected basis $w_p(n)$ is the orthogonality condition expressed as follows:

$$\frac{1}{N} \sum_{n=-(N-1)}^{0} w_p(n) \, w_{p'}(n) = \beta_p \cdot \delta_{p-p'} \tag{17}$$

where $\beta_p > 0$. This will be a good property because this will allow to decompose easily the square operation in (16).

Now, applying this orthogonality concept shown in (17), we can expand the square operation of (16) without cross terms between the time basis as:

$$\Phi_2^{UV} = C_3 + N \sum_{k=1}^{K} \ln \Phi_{2,k}^{UV} \tag{18}$$

where the k-th contribution $\Phi_{2,k}^{UV}$ can be expressed as

$$\Phi_{2,k}^{UV} = \overline{t_k^2} + \sum_{p=0}^{P-1} \beta_p \gamma_{k,p}^2(\mathbf{x},\mathbf{u}) - 2 \sum_{p=0}^{P-1} \beta_p \gamma_{k,p}(\mathbf{x},\mathbf{u}) \, \bar{t}_{k,p} \tag{19}$$

and where

$$\overline{t_k^2} = \frac{1}{N} \sum_{n=-(N-1)}^{0} t_{k,n}^2 \qquad \bar{t}_{k,p} = \frac{1}{\beta_p} \frac{1}{N} \sum_{n=-(N-1)}^{0} t_{k,n} w_p(n) \, . \tag{20}$$

Looking at the definition of these terms, we can easily note that $\overline{t_k^2}$ is the mean of the squared measurements of the k-th source and $\bar{t}_{k,p}$ are the P normalized means of the k-th source measurements thought the P time-windows $w_p(n)$.

Now rearranging terms, the general negative log-ML function presented in (18) and (19) can be expressed as:

$$\Phi_2^{UV} = C_3 + N \sum_{k=1}^{K} \ln \left[\overline{t_k^2} - \sum_{p=0}^{P-1} \beta_p \bar{t}_{k,p}^2 + \sum_{p=0}^{P-1} \beta_p \left(\bar{t}_{k,p} - \gamma_{k,p}(\mathbf{x},\mathbf{u}) \right)^2 \right] \tag{21}$$

where the term $\widehat{\widehat{\sigma}}_k^2 = \overline{t_k^2} - \sum_{p=0}^{P-1} \beta_p \bar{t}_{k,p}^2$ has been intentionally grouped and it can be understood as an estimate of the k-th variance. Using this compact notation we have:

$$\Phi_2^{UV} = C_3 + N \sum_k \ln \left[\widehat{\widehat{\sigma}}_k^2 + \sum_{p=0}^{P-1} \beta_p \left(\bar{t}_{k,p} - \gamma_{k,p}(\mathbf{x},\mathbf{u}) \right)^2 \right] \tag{22}$$

$$\Phi_2^{UV} = C_4 + N \sum_k \ln \left[1 + \left(\widehat{\widehat{\sigma}}_k^2 \right)^{-1} \sum_{p=0}^{P-1} \beta_p \left(\bar{t}_{k,p} - \gamma_{k,p}(\mathbf{x},\mathbf{u}) \right)^2 \right] \tag{23}$$

It can be demonstrated under the approximation shown in (12) that:

$$\lim_{\substack{in\ probability \\ N \to \infty}} \left(\widehat{\widehat{\sigma}}_k^2 \right)^{-1} \sum_{p=0}^{P-1} \beta_p \left(\bar{t}_{k,p} - \gamma_{k,p}(\mathbf{x},\mathbf{u}) \right)^2 = 0 \ \forall k \tag{24}$$

Then, using the approximation $ln(1+z) \approx z$ for $|z| << 1$, we can approximate Φ_2^{UK} as:

$$\Phi_2^{UV} \approx C_4 + N \sum_{k=1}^{K} \left(\widehat{\widehat{\sigma}}_k^2 \right)^{-1} \sum_{p=0}^{P-1} \beta_p \left(\bar{t}_{k,p} - \gamma_{k,p}(\mathbf{x},\mathbf{u}) \right)^2 \tag{25}$$

Although this last equation is quite similar to its equivalent in the *Known Variance* case (9), the arbitrary definition of the KxP functions $\gamma_{k,p}(\mathbf{x},\mathbf{u})$ from the K functions $f_{k,n}(\mathbf{x},\mathbf{u})$ complicates the usage of the presented development. The proposed selection for the P bases functions $w_p(n)$ will allow us to formulate (25) with the same original K functions ($f_{k,n}(\mathbf{x},\mathbf{u})$) evaluated at different temporal points (subindex n).

Now, a part from the orthogonality condition (17), the proposed temporal windows are further constrained to satisfy the following property:

$$w_p(n_{p'}) = \delta_{p-p'} \quad \forall p, p' \in [0, P-1] \tag{26}$$

where n_p is a *real* (not constrained to integer) number in the same range of the integer time subindex n: $-(N-1) < n_o < 0$. Note that condition shown in (26) can be expressed in the real domain thanks to the real domain definition of $w_p(n)$ shown in (15). This condition can be graphically observed in the right side of figure (1) where it can be appreciated that there are three points where all time-vectors are zero except one which $w_p(n) = 1$.

Taking into account the condition for the time windows shown in (26) and the approximation shown in (12), we can find the definition of the P functions $\gamma_{k,p}(\mathbf{x},\mathbf{u})$ evaluating $f_{k,n}(\mathbf{x},\mathbf{u})$ in $n = n_p, \forall p$ as follows:

$$f_{k,n_p}(\mathbf{x},\mathbf{u}) \approx \sum_{p'=0}^{P-1} w_{p'}(n_p) \, \gamma_{k,p'}(\mathbf{x},\mathbf{u}) = \gamma_{k,p}(\mathbf{x},\mathbf{u}) \tag{27}$$

where again the real-domain of the subindex n in the definition of $f_{k,n}(\mathbf{x},\mathbf{u})$ in (3) and (6) allows a real definition (not constrained to integer) of all n_p. It is also clear that more sophisticated models for the mobile movement ($\mathbf{h}_n(\mathbf{x},\mathbf{u})$) will also allow this real-domain definition in the subindex n.

Now, using (27) in (25) we have

$$\Phi_2^{UV} \approx C_4 + N \sum_{k=1}^{K} \left(\widehat{\widehat{\sigma}}_k^2 \right)^{-1} \sum_{p=0}^{P-1} \beta_p \left(\bar{t}_{k,p} - f_{k,n_p}(\mathbf{x},\mathbf{u}) \right)^2 \tag{28}$$

and now, in order to obtain a formulation similar to (9) we can redefine the variance estimate to obtain

$$\Phi_2^{UV} \approx C_4 + N \sum_{p=0}^{P-1} \sum_{k=1}^{K} \left(\widehat{\widetilde{\sigma}}_{k,p}^2\right)^{-1} \left(\bar{t}_{k,p} - f_{k,n_p}(\mathbf{x},\mathbf{u})\right)^2 \quad (29)$$

where

$$\widehat{\widetilde{\sigma}}_{k,p}^2 = \frac{\widehat{\widetilde{\sigma}}_k^2}{\beta_p} \quad (30)$$

V ML estimator

This section shows the closed-form ML estimator valid for both cases: *Known* and *Unknown Variance.* The two main ideas here are first formulating both ML functions in a common way thanks to the previous development and second splitting the ML function into two parts: the non-linear relation between measurements and position, and the linear time-evolution model for the position. These two main ideas are shown in the following subsections.

V-A General ML function

From (9) and (29) both ML functions can be generically expressed as follows:

$$\Phi_G \approx C_5 + C_6 \sum_q \sum_{k=1}^{K} \left(\widetilde{\sigma}_{k,q}^2\right)^{-1} \left(\widetilde{t}_{k,q} - f_{k,\widetilde{n}_q}(\mathbf{x},\mathbf{u})\right)^2 \quad (31)$$

where in the *Known Variance* case we have:

$$\widetilde{\mathbf{n}} = [-(N-1),\cdots,0] \qquad 1 \le q \le N \quad (32)$$

$$\widetilde{\sigma}_{k,q}^2 = \sigma_k^2 \qquad \widetilde{t}_{k,q} = t_{k,q} \quad (33)$$

and in the *Unknown Variance* case we have:

$$\widetilde{\mathbf{n}} = [n_0,\cdots,n_{P-1}] \qquad 1 \le q \le P \quad (34)$$

$$\widetilde{\sigma}_{k,q}^2 = \widehat{\widetilde{\sigma}}_{k,q}^2 \qquad \widetilde{t}_{k,q} = \bar{t}_{k,q} \quad (35)$$

where $\widetilde{n}_q$ is the q-th element inside the vector $\widetilde{\mathbf{n}}$.

The conclusion of this common expression for both cases (31) is that the ML estimation of the position in the *Unknown Variance* case can be understood as a problem with known variance taking into account the following points:

- In the *Unknown Variance* (UV) case, we have P measurements associated with some concrete non-uniformly distributed time points ($\widetilde{\mathbf{n}} = [n_0,\cdots,n_{P-1}]$) instead of N uniformly distributed (in time) measurements ($\widetilde{\mathbf{n}} = [-(N-1) \le n \le 0]$) per each one of the K sources.
- In the UV case, we have the P projections $\bar{t}_{k,p}$ to the P vectors of the time basis, instead of the N original observed measurements $t_{k,n}$.
- Instead of the K known variances σ_k (common for $\forall n$), in the UV case we have the estimated variances ($\widehat{\widetilde{\sigma}}_{k,p}^2$) specific for each one of the K sources and for each one of the P instants of time.

V-B Closed-form ML estimator

It can be demonstrated from (31), that exploiting the structure of the function $f_{k,n}(\mathbf{x},\mathbf{u})$ shown in (3) and making a first-order Taylor approximation of $f_{k,n}(\mathbf{x},\mathbf{u})$ similar to [3], the minimizer of Φ_G can also be expressed as the minimizer of $\Phi_{G,2}$:

$$\widehat{\mathbf{x}},\widehat{\mathbf{u}} = \arg\min_{\mathbf{x},\mathbf{u}} \Phi_G = \arg\min_{\mathbf{x},\mathbf{u}} \Phi_{G,2} \quad (36)$$

$$\Phi_{G,2} = \sum_q \left[\widehat{\mathbf{x}}_q - \mathbf{h}_{\widetilde{n}_q}(\mathbf{x},\mathbf{u})\right]^T \widehat{\mathbf{R}}_q^{-1} \left[\widehat{\mathbf{x}}_q - \mathbf{h}_{\widetilde{n}_q}(\mathbf{x},\mathbf{u})\right] \quad (37)$$

where $\widehat{\mathbf{x}}_q$ is the ML position estimate associated with the instant of time $\widetilde{n}_q$ and it is defined as follows:

$$\widehat{\mathbf{x}}_q = \arg\min_{\mathbf{y}} \sum_{k=1}^{K} \left(\widetilde{\sigma}_{k,q}^2\right)^{-1} \left(\widetilde{t}_{k,q} - g_k(\mathbf{y})\right)^2 \quad (38)$$

and the matrix $\mathbf{R}_q = E\left[\left(\widehat{\mathbf{x}}_q - E\left[\widehat{\mathbf{x}}_q\right]\right)^T \left(\widehat{\mathbf{x}}_q - E\left[\widehat{\mathbf{x}}_q\right]\right)\right]$ can be approximated as follows:

$$\widehat{\mathbf{R}}_q = \left[\sum_{k=1}^{K} \left(\widetilde{\sigma}_{k,q}^2\right)^{-1} \nabla g_k(\widehat{\mathbf{x}}_q) \nabla^T g_k(\widehat{\mathbf{x}}_q)\right]^{-1} \quad (39)$$

where $\nabla g_k(\mathbf{y})$ is the gradient vector of $g_k(\mathbf{y})$ defined as follows:

$$\left[\nabla g_k(\mathbf{y})\right]_l = \frac{\partial g_k(\mathbf{y})}{\partial [\mathbf{y}]_l} \quad (40)$$

where $[\ ]_l$ indicates the l-th component of a vector. It is possible to compute all estimators $\widehat{\mathbf{x}}_q$ defined in (38) following the closed-form expression presented in [3].

Finally, as a case study, it will be developed the exact expressions for the linear constant speed movement case shown in (6). Under this approach, we saw that $\mathbf{h}_{\widetilde{n}_q}(\mathbf{x},\mathbf{u}) = \mathbf{x} + \widetilde{n}_q\mathbf{s}$, so from (37) $\Phi_{G,2}$ can be expressed as:

$$\Phi_{G,2} = \sum_q \left(\widehat{\mathbf{x}}_q - \mathbf{x} - \widetilde{n}_q\mathbf{s}\right)^T \widehat{\mathbf{R}}_q^{-1} \left(\widehat{\mathbf{x}}_q - \mathbf{x} - \widetilde{n}_q\mathbf{s}\right) \quad (41)$$

Now, the joint estimation of $\mathbf{x}$ and $\mathbf{s}$ is found by minimizing with respect both parameters. The equations that must be satisfied are:

$$\nabla_{\mathbf{x}}\Phi_{G,2} = \mathbf{0} \qquad \nabla_{\mathbf{s}}\Phi_{G,2} = \mathbf{0} \quad (42)$$

where the following equations can be found:

$$\sum_q \widehat{\mathbf{R}}_q^{-1} \left(\widehat{\mathbf{x}}_q - \widehat{\mathbf{x}} - \widetilde{n}_q\widehat{\mathbf{s}}\right) = \mathbf{0} \quad (43)$$

$$\sum_q \widetilde{n}_q \widehat{\mathbf{R}}_q^{-1} \left(\widehat{\mathbf{x}}_q - \widehat{\mathbf{x}} - \widetilde{n}_q\widehat{\mathbf{s}}\right) = \mathbf{0} \quad (44)$$

From this last two expressions we can obtain two equations relating both estimates $\mathbf{x}$ and $\widehat{\mathbf{s}}$:

$$\overline{\overline{\widehat{\mathbf{x}}}}_0 = \overline{\overline{\widehat{\mathbf{R}}}}_0 \cdot \widehat{\mathbf{x}} + \overline{\overline{\widehat{\mathbf{R}}}}_1 \cdot \widehat{\mathbf{s}} \qquad \overline{\overline{\widehat{\mathbf{x}}}}_1 = \overline{\overline{\widehat{\mathbf{R}}}}_1 \cdot \widehat{\mathbf{x}} + \overline{\overline{\widehat{\mathbf{R}}}}_2 \cdot \widehat{\mathbf{s}} \quad (45)$$

where

$$\overline{\overline{\widehat{\mathbf{R}}}}_l = \sum_q \widetilde{n}_q^l \cdot \widehat{\mathbf{R}}_q^{-1} \ \ \textit{and} \ \ \overline{\overline{\widehat{\mathbf{x}}}}_l = \sum_q \widetilde{n}_q^l \cdot \widehat{\mathbf{R}}_q^{-1} \cdot \widehat{\mathbf{x}}_q \quad (46)$$

Now from (45), it is not difficult to find both closed-form estimators:

$$\widehat{\mathbf{x}} = \left(\overline{\overline{\widehat{\mathbf{R}}}}_1^{-1} \cdot \overline{\overline{\widehat{\mathbf{R}}}}_0 - \overline{\overline{\widehat{\mathbf{R}}}}_2^{-1} \cdot \overline{\overline{\widehat{\mathbf{R}}}}_1\right)^{-1} \left(\overline{\overline{\widehat{\mathbf{R}}}}_1^{-1} \cdot \overline{\overline{\widehat{\mathbf{x}}}}_0 - \overline{\overline{\widehat{\mathbf{R}}}}_2^{-1} \cdot \overline{\overline{\widehat{\mathbf{x}}}}_1\right) \quad (47)$$

$$\widehat{\mathbf{s}} = \left(\overline{\overline{\widehat{\mathbf{R}}}}_0^{-1} \cdot \overline{\overline{\widehat{\mathbf{R}}}}_1 - \overline{\overline{\widehat{\mathbf{R}}}}_1^{-1} \cdot \overline{\overline{\widehat{\mathbf{R}}}}_2\right)^{-1} \left(\overline{\overline{\widehat{\mathbf{R}}}}_0^{-1} \cdot \overline{\overline{\widehat{\mathbf{x}}}}_0 - \overline{\overline{\widehat{\mathbf{R}}}}_1^{-1} \cdot \overline{\overline{\widehat{\mathbf{x}}}}_1\right) \quad (48)$$

VI Simulation results

Numerical simulations presented here compare the performances of the proposed algorithm with the CRB. The theoretical limits needed to compare the proposed algorithm can be found for TOA measurements in [4] and the equivalent expression for TDOA measurements can be obtained in a straightforward way. The expressions of these theoretical limits have not been included for space reasons.

Simulations presented have been computed in an scenario with four BSs placed (randomly) at the points (300m,300m), (250m,-450m), (-300m,400m) and (-450m,-100m) where the position of the mobile at the end of the observed window is (10m,10m). The mobile is moving with a constant speed of 30 Km/h in the 45 noth-east direction and the TDOA measurements are taken at a constant rate of $r = 3$ (samples/second). The unknown variances used in the simulations are $\sigma = [50, 10, 5]$ chosen different to force a good estimation of the variances for a correct position estimation.

In figure (2), we can see the performance analysis of the proposed algorithm for several values of the time observed window $t = N/r$ (in seconds) and for two different polynomial order $P = [2, 3]$. It can be seen that for very short observation window ($t < 20$ *seconds*), the CRB is not attained due to an incorrect estimation of the variance vector σ. It can be also observed that for large observed window $t > 200$ *secs*, the proposed algorithm does not attain the CRB. This time, this is due to the fact that the polynomial approximation for the time evolution of the TDOA measurements shown in (12) is not realistic. For the rest of cases, the algorithm attains the CRB. It can be observed that for longer observed windows, the polynomial approximation with order three is more robust than the one with order two due to a best fit of the real time evolution. Of coarse the prise is a linear increase of the complexity.

Although, the performances of the proposed algorithm depends on the specific scenario (position of the BSs and the speed and the direction of the mobile), the algorithm presents similar performance for all cases.

VII Conclusions

This paper has presented a common framework to compare the ML position estimation in an scenario where the quality of the available measurements (TOA or TDOA) is a-priori known (unrealistic but common assumption) with the new considered case where the variance of the measurement is unknown.

The development of the ML function in the Unknown Variance case is presented under the assumption that the evolution of a certain measurement (for instance the TDOA between the first and second BSs) is modelled as a polynomial time-evolution of order P. This assumption is motivated by the fact that after an average study of the time evolution of a measurement (TOA or TDOA), they can be cleary decomposed in a polynomial way.

After presenting this approximation for the Unknown Variance case, a common model for the ML function is presented for both cases. Using this common formulation, this paper presents a closed-form estimator for the mobile position. This estimator is demonstrated to be asymptotically (in time) the ML estimator. The main advantages of this estimator are:

- Lack of initialization that is very common in Taylor-based location algorithms
- A reduction of the complexity (due to the fact that $P << N$ measurements have to be processed instead of N)
- The variances are jointly estimated per each block of N samples, so the proposed algorithm is robust to very aggressive scenarios where the variances of the available measurements are very different.

Numerical simulations show that the proposed algorithm applied to the Unknown Variance case attains the theoretical limits for a big range of lengths for the time observed windows. The algorithm presents an optimum time-window length due to the fact that a small window presents a poor estimation of the Unknown Variances and a large window does not satisfy the polynomial assumption.

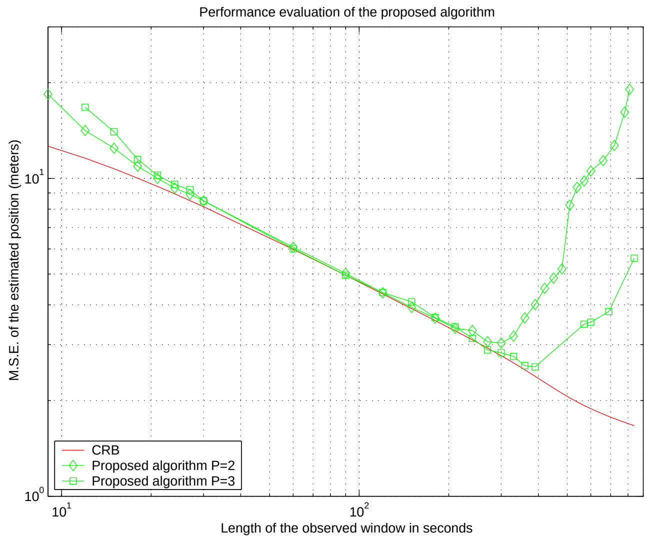

Figure 2: Performance evaluation

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