BEST QUADRATIC UNBIASED ESTIMATOR (BQUE) FOR TIMING AND FREQUENCY SYNCHRONIZATION

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ABSTRACT
This paper deals with the optimal design of quadratic Non-Data-Aided (NDA) open- and closed-loop estimators. The new approach supplies the minimum variance, unbiased NDA quadratic estimators, without the need of assuming a given statistics for the nuisance parameters, that is, avoiding the common adoption of the gaussian assumption, which does not apply in digital communications. Alternatively, if the unbiased constraint is relaxed, a bayesian open-loop estimator is presented and its performance compared with the open-loop BQUE solution. On the other hand, the closed-loop BQUE is developed, showing that it outperforms the well-known 'ad hoc' Gaussian Stochastic Maximum Likelihood (GSML) scheme for short observation windows, that is, for low-complexity implementations, only converging to the UCRBG asymptotically. Finally, the quadratic analysis is naturally extended to higher-order techniques which exhibit a better performance for high SNRs.

1. INTRODUCTION
Non-Data-Aided (NDA) synchronization has received lately a lot of attention because it simplifies the system protocols and makes unnecessary the transmission of training sequences (preambles) that reduce significantly the spectral efficiency.

So far, most algorithms have been devised by heuristic reasoning. In [1] [2] the authors presented a general framework that allowed the formulation of any NDA synchronizer based on second order moments under a Maximum Likelihood (ML) perspective. There are two basic reasons for limiting the analysis to quadratic synchronizers. It is shown [1] that the stochastic ML solution becomes quadratic for low SNRs, being still unknown for moderate or high SNRs because the difficult treatment of the unknown transmitted symbols and, moreover, they allow efficient digital implementations.

In this paper, we propose a totally different approach to the design of NDA discriminators that does not have to cope with the symbols extraction problem as in [1]. Making use of basic concepts from the estimation theory [3], we have deduced the Best Quadratic Unbiased Estimator (BQUE), that is, an estimator of the desired parameter that is quadratic, unbiased and has minimum variance. Its name has been chosen by analogy with the classical Best Linear Unbiased Estimator (BLUE) [3]. The BQUE approach allows to unify the design of open- and closed-loop synchronizers by constraining the value and/or slope of certain points of the S-curve and optimizing its performance within the expected operation range.

The structure of the paper is the following. The signal model and the problem statement are presented in Section 2. Section 3 introduces the algebraic notation used through the paper. Section 4 deduces the exact solution to the open-loop BQUE and closed expressions are obtained with the help of a discrete approximation. In Section 5 the non-bias constraint is lifted and an alternative open-loop bayesian discriminator having minimum mean squared error is presented. Section 6 deduces the closed-loop BQUE for tracking. Section 7 compares the BQUE and Gaussian Stochastic Maximum Likelihood (GSML) feedback detectors with the Gaussian Unconditional Cramer-Rao Bound (URBG). Section 8 extends the results to higher-order estimators. Simulations results and their comments can be found in Section 9 and, finally, conclusions are drawn in Section 10.

2. DISCRETE-TIME SIGNAL MODEL
A lot of problems in the signal processing field can be unified using the following signal model:

\[ \mathbf{r} = \mathbf{A}_\lambda \cdot \mathbf{x} + \mathbf{w} \] (1)

where \( \mathbf{r} \) is a vector containing \( N \) samples of the received signal (\( N_s \) samples per symbol), \( \lambda \) is the parameter to estimate (for instance, the timing error or frequency error) embedded in the transfer matrix \( \mathbf{A}_\lambda \), \( \mathbf{x} \) is the vector of transmitted symbols (unknown in a NDA scheme), \( \mathbf{w} \) is the vector of noise samples with covariance matrix \( \mathbf{R}_w = \mathbf{E} \{ \mathbf{w} \mathbf{w}^H \} \).

3. NOTATION
The following notation is introduced here to facilitate the deduction of the proposed BQUE estimators:

\[ \hat{\mathbf{R}}_\lambda = \mathbf{r} \mathbf{r}^H \]
\[ \mathbf{R}_\lambda = \mathbf{E} \{ \hat{\mathbf{R}}_\lambda \} = \mathbf{A}_\lambda \mathbf{A}_\lambda^H + \mathbf{R}_w \]
\[ \mathbf{S}_\lambda = \frac{1}{N_s} \mathbf{R}_\lambda = \mathbf{D}_\lambda \mathbf{A}_\lambda \mathbf{A}_\lambda^H + \mathbf{A}_\lambda \mathbf{D}_\lambda \mathbf{A}_\lambda^H ; \quad \mathbf{D}_\lambda = \frac{N_s}{\lambda} \mathbf{A}_\lambda \]
\[ \hat{\mathbf{R}}_\lambda = \text{vec} (\hat{\mathbf{R}}_\lambda) ; \quad \mathbf{R}_\lambda = \text{vec} (\mathbf{R}_\lambda) ; \quad \mathbf{S}_\lambda = \text{vec} (\mathbf{S}_\lambda) \] (2)
where $R_{\lambda}$ is the sample covariance matrix and the operator $\text{vec}(M)$ stacks the columns of $M$ in the column vector $M$. Using the notation defined in the previous section, the generic equation of a quadratic discriminator is:

$$
\lambda = r^H M r = \text{Tr}(M \tilde{R}_{\lambda}) = M^H \tilde{R}_{\lambda}
$$

(3)

where $\text{Tr}()$ is the trace operator, $M$ is the matrix containing the discriminator coefficients (complex) and $M$ is defined as:

$$
M = \text{vec}(M^H)
$$

(4)

4. OPEN-LOOP (OL) BEST QUADRATIC UNBIASED ESTIMATOR (OL-BQUE)

The minimum variance unbiased $(MVE)$ estimator $\hat{M}$ if the received parameter is $\lambda$ has this expression:

$$
\hat{M} = \arg\min_M \{ \| \lambda - M \|_F^2 \}
$$

(5)

subject to $M^H \tilde{R}_{\lambda} = \lambda$ and, thus, the cost function to minimize is the following:

$$
C(\lambda) = M^H Q_{\lambda} M + (M^H \tilde{R}_{\lambda} - \lambda) \tilde{\alpha}_{\lambda}
$$

(6)

where $Q_{\lambda} = E \{ \tilde{R}_{\lambda} \tilde{R}_{\lambda}^H \}$ and the Lagrange multiplier $\tilde{\alpha}_{\lambda}$ impose the non-bias constraint $M^H \tilde{R}_{\lambda} = \lambda$. The fourth-order moments contained in $Q_{\lambda}$ can be computed as follows for any symmetric constellation $E\{x_n^2\} = 0$ if $n$ is odd) with uncorrelated and identically distributed symbols $E\{x_n x_n^j\} = E\{x_n x_n\} = 0$ if $n \neq j$:

$$
Q_{\lambda}(N + j, Nk + l) = E\{r_j r_k x_n^i\} = R_{ij} R_{kl} + R_{ik} R_{jl} +
+ (\mu_1 - 2\mu_2) \cdot (a_i \circ a_j) (a_k \circ a_l)^H
$$

(7)

$+$

$$
(\mu_1 - 2\mu_2) (a_i \circ a_j) (a_k \circ a_j)^H - (a_i \circ a_j \circ a_k \circ a_l) (a_j \circ a_l)^H
$$

where $R_{ij}$ is the element $(i,j)$ of $R_{\lambda}$, $a_k$ the $n$-th row of $A_{\lambda}$, $\mu_2 = E\{\{x_n\}^2\}$ and $\mu_3 = E\{\{x_n\}^3\}$ if $n = N$ the $n$-th element of the row vector $p$ and $\circ$ stands for the Hadamard product of matrices.

Any statistical a priori knowledge of the parameter of interest $\lambda \in \Lambda = \{ |\lambda| \leq \Delta_{\lambda} \}$ can be introduced in the optimization process by means of a bayesian approach, that is, by averaging the cost function in (6) with respect to the adopted prior:

$$
C_{alt} = E_{\lambda} \{ C(\lambda) \} = \int_{\Lambda} C(\lambda) W_{\lambda} d\lambda \equiv \int_{\Lambda} \{ M^H Q_{\lambda} M + M^H R_{\lambda} \alpha_{\lambda} \} d\lambda
$$

(8)

where $\alpha_{\lambda} = W_{\lambda} \tilde{\alpha}_{\lambda}$ and all the irrelevant terms have been wiped out from the last equation. The weighting function $W_{\lambda} = f_{\lambda}(\lambda)$ is the prior of the parameter of interest. If no a priori knowledge is available, a uniform prior shall be adopted within the operative range $\Lambda$, that is, $W_{\lambda} = \frac{1}{\Delta_{\lambda}}$.

The solution of (8) has the following expression:

$$
M = Q^{-1} \tilde{R}
$$

(9)

with $Q = \int_{\Lambda} Q_{\lambda} W_{\lambda} d\lambda$ and $\tilde{R} = \int_{\Lambda} R_{\lambda} \alpha_{\lambda} d\lambda$. The value of the multipliers $\alpha_{\lambda}$ $(\forall \lambda \in \Lambda)$ that force the unbiased solution within the whole interval $\Lambda$ requires the solution to the following system of integral equations:

$$
M^H R_{\lambda} = \tilde{R}_{\lambda} = \int_{\Lambda} R_{\lambda}^H \alpha_{\lambda} d\lambda \cdot Q^{-1} R_{\lambda} = \lambda
$$

(10)

At that point, we have opted for a discrete approximation of the integral in (10) considering only a finite set of constraints $\lambda_k = \{\lambda_1, \ldots, \lambda_N\}$. Thus, we obtain that $R_{\lambda} = R_{\lambda k} \alpha$ and (10) becomes $R_{\lambda k}^H Q^{-1} R_{\lambda k} = \lambda_k$ (after some manipulations) incorporating the definitions below:

$$
R_{\lambda} = [R_{\lambda 1}, \ldots, R_{\lambda N}] \quad \alpha = [\alpha_1, \ldots, \alpha_N]^T
$$

(11)

The discrete approximation of the solution is therefore:

$$
M = Q^{-1} (R_{\lambda}^H \alpha) = Q^{-1} R_{\lambda}^H R_{\lambda}^H Q^{-1} R_{\lambda}^H \alpha
$$

(12)

It turns out that the matrix $R_{\lambda k}^H Q^{-1} R_{\lambda k}$ in (12) can be singular if the oversampling $(N_s)$ and the length of $r$ are not large enough. In that case, the set of constraints $\alpha$ cannot be exactly fulfilled and the pseudo-inverse operator $(\cdot)^+$ will supply the least-squares fitting.

5. UNCONSTRAINED OPEN-LOOP BAYESIAN ESTIMATOR (OL-BAYES)

In this section we present an alternative criterion to design open-loop estimators from a bayesian approach when relaxing the unbiased constraint. The discriminator coefficients $M$ will be those that minimize the following cost function:

$$
C_{alt} = E_{\lambda} \{ \| M^H \tilde{R} - \lambda \|_F^2 \} \equiv \int_{\Lambda} \{ M^H Q_{\lambda} M - M^H \tilde{R} \} d\lambda
$$

(13)

where the last equivalence only conserves the terms dependent on $M^H$ and where $\tilde{R} = \int_{\Lambda} R_{\lambda} W_{\lambda} d\lambda$ and $Q$ was introduced in (8).

The expression of the discriminator $M$ minimizing (14) is therefore:

$$
M = Q^{-1} \tilde{R}
$$

(14)

and both $\tilde{R}$ and $Q$ admit analytical solutions for uniform priors, i.e., $W_{\lambda} = \frac{1}{2\Delta_{\lambda}}$.

6. CLOSED-LOOP (CL) BEST QUADRATIC UNBIASED ESTIMATOR (CL-BQUE)

In this section the estimator is required to track the parameter fluctuations around a mean with minimum variance for a given loop bandwidth, i.e., for a given value of the S-curve slope at $\lambda=0$. The pretended BQUE discriminator of the parameter errors will be that minimizing the following cost function:

$$
C_{alt} = M^H Q_{0} M + M^H R_{0} \alpha_0 + \left( M^H S_{0} - 1 \right) \beta_0
$$

(15)

where the Lagrange multipliers $\alpha_0$ and $\beta_0$ impose the non-bias constraints $M^H R_{0} = 0$ and $M^H S_{0} = 1$ at $\lambda=0$.

It can be shown that the constraint $\alpha_0$ is always fitted due to the discriminator symmetry and the CL – BQUE solution reduces to:

$$
M = \beta_0 \cdot Q_0^{-1} S_0 = \frac{Q_0^{-1} S_0}{S_0^H Q_0^{-1} S_0} R_{0}^H \alpha_0
$$

(16)

and its tracking error variance is therefore

$$
\sigma_\lambda^2 = \frac{1}{S_0^H Q_0^{-1} S_0}
$$

(17)
Figure 1: Normalized timing variance ($\sigma_\phi^2/T^2$) for the GSML and CL–BQUE discriminators $N=4$. The fourth-order discriminator proposed in reference [4] is also plotted and compared with the MCRB [5].

7. CL-BQUE VS. MAXIMUM LIKELIHOOD APPROACH

Other suitable approach to the problem is to treat the received data ($r$) as if they were gaussian. The resulting discriminator, called Gaussian Stochastic Maximum Likelihood (GSML), has the following structure in the uniparametric case [2]:

$$\hat{\lambda} = \frac{Tr(R_0^{-1}S_0 R_\lambda^{-1} \hat{R}_\lambda)}{Tr[R_\lambda^{-1}S_0]}$$  \hspace{1cm} (18)

This discriminator is known to attain the Gaussian Unconditional Cramer–Rao Bound (UCRBG) if $N \to \infty$ [1].

In Section 9 simulations have shown that the CL–BQUE (Section 6) becomes asymptotically equivalent to the GSML and, hence, both attain the UCRBG (asymptotically). However, when the length of $r$ is short, the variance of the CL–BQUE is below that of the GSML and the UCRBG is not attained. This fact confirms the UCRBG as a suitable bound for quadratic (unbiased) NDA discriminators but it also proves that it can not be attained if the observation window is not large enough. In that case, the performance of the CL–BQUE (17) can be used as a tighter bound valid for any quadratic unbiased NDA discriminator.

8. EXTENSION TO HIGHER-ORDER DISCRIMINATORS

In this section the procedure for designing optimal open- and closed-loop estimators is extended to $n$-order discriminators (with $n > 2$).

For the $n$-order case, equation (3) can be rewritten as follows:

$$\hat{\lambda} = M^H \hat{R}^{(n)}$$  \hspace{1cm} (19)

where

$$\hat{R}^{(n)} = \hat{R} \otimes \hat{R}^{(n-2)} \quad n > 2 \text{ (even)}$$  \hspace{1cm} (20)

and $\otimes$ stands for the Kronecker product of matrices.

Odd values of $n$ are not considered because all modulation schemes in practice are symmetric with respect to the origin and so the odd moments are always null.

All the expressions included in previous sections are then usable if the following substitutions are done previously:

$$\hat{R} \rightarrow \hat{R}^{(n)}$$
$$\hat{R}_\lambda \rightarrow \hat{R}_\lambda^{(n)} = E \left\{ \hat{R}^{(n)} \right\}$$
$$\hat{S}_\lambda \rightarrow S_\lambda^{(n)} = \frac{1}{M} \hat{R}_\lambda^{(n)}$$
$$Q_\lambda \rightarrow Q_\lambda^{(n)} = E \left\{ R_\lambda^{(n)} \left( R_\lambda^{(n)} \right)^T \right\}$$  \hspace{1cm} (21)

Generally, the complexity and the minor improvement reflected in the system BER with respect to quadratic algorithms, do not justify the utilization of higher-order synchronizers in communications systems. However, when the purpose is not strictly synchronization, but the exact estimation and/or tracking of the time of arrival and/or the Doppler offset of the incoming signal, which is the case of advanced navigation and positioning systems (DGPS, GNSS, etc.), they could be taken into account despite their complexity. In any case, the higher-order study herein is valuable because it supplies new bounds that give information on the potential improvement these techniques can yield (Figure 1).

9. SIMULATION RESULTS

The simulations have been carried out for the MSK (Minimum Shift Keying) modulation as a particular case of the binary Continuous Phase Modulations CPM. This transmission scheme is adopted because it allows a simple extension to any linear digital modulation and to any multiple access modulation, as well [6]. Recall that the Laurent expansion [7] [5] allows the formulation of binary CPM signals in terms of the model presented in Section 2. The simulations have been done for additive white gaussian noise (AWGN) and two samples per symbol ($N_{sa} = 2$).
- Figure 1: the GSML and CL-BQUE are compared with the UCRBG when the vector of samples r is short (N=4). It is clear that in this case the variance of both discriminators is above the UCRBG. The discrepancies are more important for high EbNo because the gaussian assumption becomes more exact as the EbNo is reduced. Figure 1 also shows how fourth-order detectors [4] are capable to be below the UCRBG and nearer the DA performance. It is also remarkable that for low SNRs the UCRBG becomes a valid bound for the variance of any NDA detector irrespective of its order.

- Figure 2: the asymptotic convergence of the CL-BQUE and the GSML is shown. The variance depicted in the figure is for an open-loop configuration when λ ≈ 0 (also Fig. 4). The corresponding closed-loop tracking variance is approximately L0 = 0.5/Bs,T times lower if the normalized loop bandwidth (B,T) is very small and L0 \gg N.

- Figure 3 and 4: the two open-loop schemes proposed in the paper (Sections 4 and 5) are compared. Figure 4 shows how the OL-BAYES can reduce the mean squared error within the designed interval Δλ (even for a extremely high EbNo=40 dB) because it is not forced to be unbiased (see figure 3).

The behaviour of the studied closed-loop discriminators (GSML and CL-BQUE, Sec. 6) for λ ≠ 0 is also depicted. Figures 3 and 4 show their specialization for λ = 0 (tracking). It is also noteworthy that the CL-BQUE has a better behaviour than the GSML outside the steady-state situation (λ ≠ 0).

10. CONCLUSIONS
This paper presented a new, versatile approach for designing both open- and closed-loop optimal synchronizers with constraints on the S-curve shape (non-bias restrictions). If a little amount of bias is tolerated, a very simple, elegant bayesian estimator was formulated in Sec. 5 which is found to reduce the mean squared error within the designed range of the parameter.

For the closed-loop case, an optimal (unbiased) parameter error detector is obtained whose tracking variance is a lower bound for any NDA quadratic unbiased discriminator with independence of the number of samples it processes. A comparison with the classical GSML is carried out and their asymptotic convergence proved empirically. However, the BQUE is observed to outperform the GSML for short data vectors and high SNRs.

Finally, the formulation of the paper is extended to higher-order synchronizers and their utilization discussed.

11. REFERENCES


